Recall \( M \times N \) 0,1 matrix \( \Phi \) has SCC \( f : [N]^2 \rightarrow N \)
if \( \forall \) submatrices \( \Phi^* \) of \( \Phi \) w.r.t. \( n \in N \) columns
there are at most \( f(n, k) \) cells of depth \( \leq k \), \( \forall n \leq n \).

\[
\min \{ c^T x : \Phi x \geq 1, x \geq 0, x \text{ int} \}
\]

Thm 1: If \( \Phi \) has SCC \( f(n, k) = n \phi(n) k^c \)
the \( \exists \) rand.
\( O(\max \{1, \log \phi(n)\}) \sim \Phi \).
\( \leq c(n) \)

\textbf{Last time:} Sufficient to resolve the following Q.

\textbf{Give:} \( M \times N \) 0,1-matrix \( \Phi^* \) w.r.t.
- SCC \( f(n, k) = n \phi(n) k^c \)
- each cell is \( L = M/2 \) deep
  (i.e., \( \geq L \) 1's in each row)

\textbf{Find:} \( \mathcal{C} \subseteq [N] \) feasible cover s.t.
that \( \forall j \in \mathcal{C} \) :
\( Pr[j \in \mathcal{C}] = O\left(\frac{\phi(n)}{L}\right) \)

\textbf{Overview} algorithm works in phases
In a phase we are given a
**Overview**

**Algorithm covers in phase**

In a phase we are given a maxn submatrix $F$ of $A^*$ of depth $\geq K$

1. **Terminate if $K$ small**

\[
\left[ \max \{\log k, c(n)\} \geq \frac{k}{12(c+3)} \right]
\]

$\Rightarrow$ **force all remaining columns into the cover**

2. **Otherwise**

\[
\text{Comp: pol} \text{-hin } [n] = F \cup RUF <-
\]

$\downarrow$ forced $\uparrow$ rejected

Submatrix $B$ is obtained from $F$ by dropping

- Columns in $RUF$
- now that there 1 in at least one $F$ col.

**Want:** $B$ is at least $k/2$-deep

$\Rightarrow$ on to next phase till $B$ and $k/2$

**Final Cor:** all forced columns over all situations
Phase Details

1. Mark each column \( j \) of \( \Pi \) with prob \( \frac{1}{2} + h(N,K) \) if \( \Pi(j) < \frac{k}{2} \) is in \( T \) columns, force a column \( j \) \& \( T \) that has 1 in row \( i \) (\( j \) covers row \( i \)).

Want to achieve

(i) \( B \) is \( k_2 \)-depr \( \checkmark \)

(ii) Each col \( j \) not in \( T \) will prob \( \leq \frac{1}{2} + h(N,K) \) \( \checkmark \)

(iii) Each column of \( \Pi \) is in \( F \) with prob \( \leq \frac{1}{k^2} \)

Will ensure (iii) by choosing \( h(N,K) \) just right.

Note:

- Increasing \( h(N,K) \) decreases forcing probability in non-terminal phase
- Increasing \( h(N,K) \) increases
• increasing $h(m,k)$ increases prod. that a column is retained

$\Rightarrow$ and hence forcing prod in terminal phase

turns out $h(n,k) = \sqrt{\frac{\log k + \phi(n)}{k}}$ words.

**How to ‘force’ the right columns**

A $mn \times m$ matrix

Call $R \subseteq [m]$ a $k$-cluster if $\exists C \subseteq [n], |C| = k$ and submatrix of $A$ induced by $R, C$ is all-1s.

ex:

$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ * forms a 2-cluster

**Key Lemma** $A$ $mn \times n$, $k$-deep matrix,

$SCC \downarrow (m,k) = n \phi(n) k^C$

Thus is an efficiently computable map

$\downarrow : [m] \rightarrow [n]$

s.t.
• Column \( \delta(i) \) covers row \( i \)
  (has a one in row \( i \)) \( \begin{align*}
i : \delta(i) &= 1^3
\end{align*} \)

• The pre-image \( \delta^{-1}(j) \) of column \( j \)
can be partitioned into \( \phi(n) k^{c+1} \) \( k \)-clusters

**Proof Sketch**
Suppose \( \Phi \) is \( k \)-deep and has SEC \( \gamma(n,k) = n \phi(n) k^c \)

\[ \Rightarrow \exists \text{col. } j \text{ that has } 1 \text{ in at most } \phi(n) k^{c+1} \text{ cell of depth exactly } k \]

**Why?**
removing rows of depth \( > k \) and duplicate row \( \Rightarrow \Phi' \)

\( \Phi' \) has \( \leq n \phi(n) k^c \) rows with \( k \)’s and

\[ \Rightarrow n \phi(n) k^{c+1} \text{ } 1’s \text{ in matrix} \]

\[ \Rightarrow \exists \text{col. that has } \leq \phi(n) k^{c+1} \text{ is.} \]

**Algorithm to construct \( \Phi \)**

While \( \Phi \) has rows left

• pick col. \( j \) as above that intersects at most \( \phi(n) k^{c+1} \) cells of depth \( k \)
• assign these cells to \( j \)
• delete from \( \Phi \) the column \( j \) and all rows belonging to cells just assigned.

[End while]
(Note: if \( \theta \) has depth \( > k \), then we may drop an arbitrary column.)

**Lemma 1** In a non-terminating phase of algo, a column \( j \) of \( \theta \) is forced if \( \text{pred} \leq \frac{1}{k^2} \).

**Proof:** Consider \( j \in [n] \). By key lema, can partition \( \mathscr{H}^{-1}(j) \) into \( \phi(n)k^{c+1} \)-clusters \( R_1, \ldots, R_p \).

Now: \( j \) is forced if row in one of the clusters are \( < k^2 \)-deep

**Pide R_i:** Let \( C \subseteq [n] \) be the \( k \) columns defining cluster \( R_i \).

Define \( \nu = \# \text{col of } C \text{ marked} \)

\[ E[\nu] = k \cdot \left( \frac{1}{2} + h(N,k) \right) = \mu \]

\( Z \): sum of indep. ind. dist. 0,1-\( \nu \)

\( \text{Chernoff bound: sd. prd. of deviation of } Z \text{ from } \mu \)
\[ \Pr[Z \leq (1-e)\mu] \leq e^{-e^2\mu/3} \quad (e \in (0,1)) \]

\( j \) is forced d/c cluster \( R_i \) if fewer than \( k/2 \) columns in \( C \) are marked:

\[ \Pr[Z \leq k/2] = \Pr[Z \leq (1 - \frac{k h(N,k)}{k/2 + k h(N,k)}) \mu] \]

\[ \leq e^{-\frac{1}{3} \left( \frac{k h(N,k)}{k/2 + k h(N,k)} \right)^2 \left( \frac{k}{2} + k h(N,k) \right)} \]

\[ = e^{-\frac{1}{3} \left( \frac{k h(N,k)}{\frac{k}{2} + h(N,k)} \right)^2} \leq e^{-\frac{2}{3} k h(N,k)^2} \]

Union bound \( \Rightarrow \) \( \Pr \) that \( j \) is forced

\[ \leq \phi(n) k^{c+1} \cdot e^{-\frac{2}{3} k h(N,k)^2} \]

new pride: \( h(N,k) = \sqrt{\frac{1}{2} \left( \frac{c+3}{c-1} \log k + \epsilon k \right)} \leq \log \phi(n) \)

\[ \leq \frac{1}{k^2} \]

So: marking prod. is large enough to ensure that in each phase, forcing prod is relatively low.
Lemma 2. After \( t \) phases any column \( j \in [N] \) still remains with prob \( \frac{\text{oc}(t)}{2^t} \)

**Proof:** \( j \) remains if it is marked in each phase. Happens with prob

\[
\Pr_e = \prod_{i=0}^{t-1} \left( \frac{1}{2} + h(N, \frac{L}{2^i}) \right)
\]

\[
h(N, \frac{L}{2^i}) = \sqrt{\frac{N}{2} \frac{(c+1) \log \frac{4L}{i} + \text{oc}(N)}{L/2^i}} = O(1) \sqrt{\frac{\log \left( \frac{L}{i} + \text{oc}(N) \right)}{L}} 2^i
\]

\[
= \prod_{i=0}^{t-1} \left( \frac{1}{2} + o(1) \sqrt{2^i \frac{\log \left( \frac{L}{i} + \text{oc}(N) \right)}{L}} \right)
\]

Suppose termination condition does not hold at any phase \( t \):

\[
\text{oc}(N) < \frac{L/2^t}{12(c+3)} \quad \text{depth 16 in phase } t
\]

\[
\leq \prod_{i=0}^{t-1} \left( \frac{1}{2} + o(1) \sqrt{2^i \frac{\log (L-i)}{L}} + o(1) \right)
\]

\[
= \frac{O(1)}{2^t} \quad \square
\]

*missing details in paper [CGKS'12]*
**Note:**

\( \log k \geq \frac{k}{12(c+3)} \) or \( e(n) \geq \frac{k}{12(c+3)} \)

\[ \implies \log k + e(n) \geq \frac{k}{12(c+3)} \]

\[ \implies k \leq O(1) e(n) \]

\[ \implies \text{need at least } O(1) \log \left( \frac{L}{e(n)} \right) \]

phase to read termination

\[ \implies \text{Rob that a column } j \text{ is retained through all iterations} \]

\[ \leq \frac{O(1)}{2 \log L/e(n)} \]

\[ = O\left( \frac{e(n)}{L} \right) \]

**Lemma:**

A column is forced in any phase with prob \( O\left( \frac{e(n)}{L} \right) \).

\[ \implies \text{show this for terminating phase.} \]

Suffices to consider non-terminating phases.

In phase \( i \), depth is \( k = \frac{L}{2^i} \). Forcing prob for col. \( j \) is at most

\[ \frac{O(1)}{2^i} \cdot \frac{1}{k^2} = \frac{O(1)}{2^i} \left( \frac{2^i}{L} \right)^2 = O\left( \frac{2^i}{L^2} \right) \]
pr. that jurying
i survives in i
to i

Sum over all non-terminating phases:

\[
\frac{o(1)}{L} \sum_{i=0}^{\log L} \frac{2^i}{L} = \frac{o(1)}{L}
\]

The lemma completes pf. of main theorem.