CR-Cover (4)

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Recall $M \times N$ 0,1 matrix A has SCC $f : [N]^2 \to \mathbb{N}$ (non-decr. in both params)
if $\forall$ submatrices A^* of A with $n \le N$ columns
there are at most $f(n,k)$ cells of depth $\le k$, $\forall k \le n$.

(SC) $\min \{ c^T x : Ax \ge \mathbb{1}, x \ge 0, x \text{ int} \}$

Thm If A has SCC $f(n,k) = n\,\phi(n)\,k^c$ (constant; ϕ non-decr. in n)
then $\exists$ rand.
$O(\underbrace{\max\{1, \log \phi(n)\}}_{\ell(n)})$-apx.

Last time: Suffices to resolve the following Q.

Given: $M \times N$ 0,1-matrix A^* with
- SCC $f(n,k) = n\,\phi(n)\,k^c$
- each cell is $L = N/2$ deep
  (i.e.: $\ge L$ 1's in each row)

Find: $\mathcal{C} \subseteq [N]$ feasible cover such
that $\forall j \in [N]$:
$$\Pr[j \in \mathcal{C}] = O\left(\frac{\ell(N)}{L}\right)$$

Overview algorithm works in phases
In a phase we are given a

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In a phase we are given a $m \times n$ submatrix A of A^* of depth $\geq K$

① Terminate if K small

$$\left[\max\{\log K, \ell(w)\} \geq K/12(c+3) \right]$$

$\Longrightarrow$ force all remaining columns into the cover

② otherwise

Comp: partition $[n] = F \cup R \cup T$

forced $\rightarrow F$, rejected $\rightarrow R$

Submatrix B is obtained from A by dropping

- Columns in $R \cup F$
- rows that have 1 in at least one F col.

Want: B is at least $K/2$ -deep

$\Longrightarrow$ on to next phase with B and $K/2$

Final Cover: all forced Columns over all iterations

iterations

[Phase Details]

ⓐ mark each column j of A with prob $\frac{1}{2} + h(N,K)$ ← phase dependent, ≤ 1 in non-term phases

T: all marked columns

ⓑ for each row i with $< \frac{K}{2}$ 1s in T columns,
force a column $j \notin T$ that has 1 in row i
(j covers row i)

Want to achieve

(i) B is $K/2$ -deep ✓

(ii) Each col of A md. with prob $\le \frac{1}{2} + h(N,K)$ ✓

(iii) Each column of A is in F with prob $\le \frac{1}{K^2}$

Will ensure (iii) by choosing $h(N,K)$ just right.

Note:

- increasing $h(N,K)$ decreases forcing probability in non-terminal phase
- increasing $h(N,K)$ increases

- increasing $h(N,k)$ increases prob. that a column is retained
  ↳ and hence forcing prob in terminal phase

turns out $h(N,k) = O\left(\sqrt{\frac{\log k + \ell(N)}{k}}\right)$

works.

How to 'force' the right columns A $m \times n$ matrix

Call $R \subseteq [m]$ a <u>k-cluster</u> if $\exists\, C \subseteq [n], |C| = k$ and submatrix of A induced by R, C is all-1s.

ex:

$$A \quad \begin{array}{c} \\ * \\ \\ * \end{array}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$*$ forms a 2-cluster

Key Lemma A $m \times n$, k-deep matrix, SCC $f(n,k) = n\,\phi(n)\,k^c$

There is an efficiently computable map
$$f : [m] \to [n]$$
s.t.

- Column $\sigma(i)$ covers row i
  (has a one in row i)

$\{i : \sigma(i) = j\}$

- The pre-image $\sigma^{-1}(j)$ of column j
  can be partitioned into
  $\phi(n) k^{c+1}$ k-clusters

Pf sketch Suppose A is k-deep & has scc $f(n,k) = n\phi(n)k^c$
$\Longrightarrow \exists$ col. j that has 1 in at most $\phi(n)k^{c+1}$ cells of depth exactly k

why? remove rows of depth $> k$ and duplicate rows $\Longrightarrow A'$

A' has $\leq n\,\phi(n)k^c$ rows with k 1's each
$\Longrightarrow n\,\phi(n)k^{c+1}$ 1's in matrix
$\Longrightarrow \exists$ col. that has $\leq \phi(n)k^{c+1}$ 1s. $\boxtimes$

Algorithm to construct σ A k-deep

While A has rows left
- pick col. j as above that intersects at most $\phi(n)k^{c+1}$ cells of depth k
- assign these cells to j
- delete from A the column j and all rows belonging to cells just assigned.

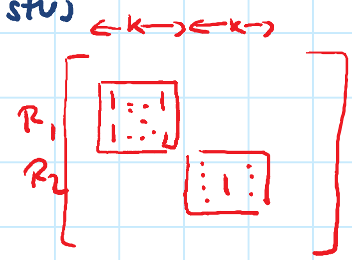
(note: if A has depth $> k$ [illegible])

(note: if A has depth $> k$ then we may drop an arbitrary col.)

Lemma 1 In a non-terminating phase of algo, a column j of A is forced with prob $\leq \frac{1}{k^2}$.

Pf: Consider $j \in [n]$. By key Lemma, can partition $f^{-1}(j)$ into $\underbrace{\phi(n) k^{c+1}}_{p}$ k-clusters

(**) $$R_1, \ldots, R_p$$



Now: j is forced if rows in one of the clusters are $< k/2$-deep

Pick R_i: let $C \subseteq [n]$ be the k columns defining cluster R_i

Define rv $Z = \#$ cols of C marked

$$E[Z] = k \cdot \left(\frac{1}{2} + h(n,k)\right) =: \mu$$

Z: sum of indep. id. dist. 0,1-rv

Chernoff bound: bd. prob. of deviation of Z from μ

from μ

$$(*) \quad \Pr[Z \le (1-\epsilon)\mu] \le e^{-\epsilon^2\mu/3}$$

$(\epsilon \in (0,1))$

j is forced b/c cluster R; if fewer than $k/2$ columns in C are marked:

$$\Pr[Z \le \tfrac{k}{2}] = \Pr\left[Z \le \left(1 - \frac{k\,h(N,k)}{\frac{k}{2} + k\,h(N,k)}\right)\mu\right]$$

$$\overset{(*)}{\le} e^{-\frac{1}{3}\left(\frac{k\,h(N,k)}{k/2 + k\,h(N,k)}\right)^2 \left(\frac{k}{2} + k\,h(N,k)\right)}$$

$$= e^{-\frac{1}{3}\frac{k\,h(N,k)^2}{\underbrace{\frac{1}{2} + h(N,k)}_{\in (\frac{1}{2},1]}}} \le e^{-\frac{2}{3}k\,h(N,k)^2}$$

Union bound $\Longrightarrow$ pr. that j is forced

$$\le \phi(n)\,k^{c+1} \cdot e^{-\frac{2}{3}k\,h(N,k)^2}$$

now pick : $h(N,k) = \sqrt{\frac{3}{2}\,\frac{(c+3)\log k + \ell(N)}{k}}$ $\;\ell(N) \ge \log \phi(N)$

$$\le \frac{1}{k^2}$$

So: marking prob. is large enough to ensure that in each phase, forcing prob is relatively low

Lemma 2 After t phases any column $j \in [N]$ still remains with prob $\frac{O(1)}{2^t}$

Pf: j remains if it is marked in each phase. Happens with prob

$$P_t = \prod_{i=0}^{t-1} \left(\frac{1}{2} + h\left(N, \frac{L}{2^i}\right)\right)$$

$$h\left(N, \frac{L}{2^i}\right) = \sqrt{\frac{3}{2} \frac{(c+3) \log L/2^i + \ell(N)}{L/2^i}} = O(1) \sqrt{\frac{\log(L) - i + \ell(N)}{L} \cdot 2^i}$$

$$= \prod_{i=0}^{t-1} \left(\frac{1}{2} + O(1) \sqrt{2^i \frac{\log L - i + \ell(N)}{L}}\right)$$

Suppose termination condition does not hold at beg of phase t:

$$\ell(N) < \frac{L/2^t}{12(c+3)} \leftarrow \text{depth ld in phase } t$$

$$\leq \prod_{i=0}^{t-1} \left(\frac{1}{2} + O(1) \sqrt{\frac{2^i \log(L-i)}{L} + \frac{O(1)}{2^{t-i}}} \right)$$

$$\rightsquigarrow = \frac{O(1)}{2^t} \qquad \square$$

missing details in paper [CGKS'12]

Note: algo terminates when either

$$\log \kappa \geq \frac{\kappa}{12(c+3)} \quad \text{or} \quad \ell(N) \geq \frac{\kappa}{12(c+3)}$$

$$\Longrightarrow \log \kappa + \ell(N) \geq \frac{\kappa}{12(c+3)}$$

$$\Longrightarrow \kappa \leq O(1)\, \ell(N)$$

$\Longrightarrow$ need at least $O(1) \log\left(\frac{L}{\ell(N)}\right)$

phases to reach termination

$\Longrightarrow$ Prob that a column j is retained through all iterations $\leq \dfrac{O(1)}{2^{\log L/\ell(N)}}$

(I)

$$= O\left(\frac{\ell(N)}{L}\right)$$

Lemma A column is forced in any phase with prob $O\left(\frac{\ell(N)}{L}\right)$.

Pf: (I) shows this for terminating phase. Suffices to consider non-terminating phases.

In phase i, depth is $\kappa = \frac{L}{2^i}$. Forcing prob for col. j is at most

$$\underbrace{\frac{O(1)}{2^i}} \cdot \underbrace{\frac{1}{\kappa^2}} = \frac{O(1)}{2^i}\left(\frac{2^i}{L}\right)^2 = O\left(\frac{2^i}{L^2}\right)$$

pr. that j survives to i

forcing in i

Sum over all non-terminating phases:

$$\frac{O(1)}{L} \sum_{i=0}^{\log L} \frac{2^i}{L} = \frac{O(1)}{L}$$

The Lemma completes pf. of main theorem.