CR-Cover (3)

Monday, June 13, 2016 1:48 PM

Recall (CIP) $\min \{c^T x : Ax \geq b,\ 0 \leq x \leq d,\ x \text{ int}\}$

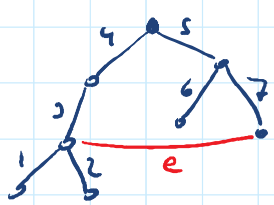
(CCIP) $\min \{c^T x : A[S]x \geq b,\ 0 \leq x \leq d,\ x \text{ int}\}$

Thm1 CCIP has $O(\gamma + \omega)$-apx if underlying 0,1-CIP and PCIP have int. gap γ and ω, respectively.

Remaining Q Suppose A is $m \times n$ 0,1 network matrix. Can CCIP be approx. well?

Def: Call A a tree-fundamental cycle (TFC) matrix if $\exists\ G = (V, E)$ and tree $T \subseteq E$ s.t. A has row for each $e \in T$ and col. for each $e \in E \setminus T$ and

$$A_{e,e'} = \begin{cases} 1 & : e \text{ on fund cycle in } T + e' \\ 0 & : \text{oth.} \end{cases}$$



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \quad e \quad \\ \left[\begin{array}{ccc} & 0 & \\ & 0 & \\ & 1 & \\ \cdots & 1 & \cdots \\ & 1 & \\ & 0 & \\ & 1 & \end{array}\right] \end{array}$$

note: every 0,1-netw. matrix is a TFC matrix (not converse)

Thm2 [Chan, Grant, K., Sharpe '12] A TFC matrix $\Rightarrow$ underlying PCIP has $O(1)$

integrality gap

$\underset{[CGK'10]}{\Longrightarrow}$ $O(1)$-apx for CCIP when $\exists$ TFC matrix

Thm 2 is proved using geometric ideas. Canonical prob:

X : collection of points in a fixed dim Euclidean space

S : collection of objects – disks, squares, half-spaces ...

Goal: find min-card (min-cost) collection $C \subseteq S$ that covers all of X

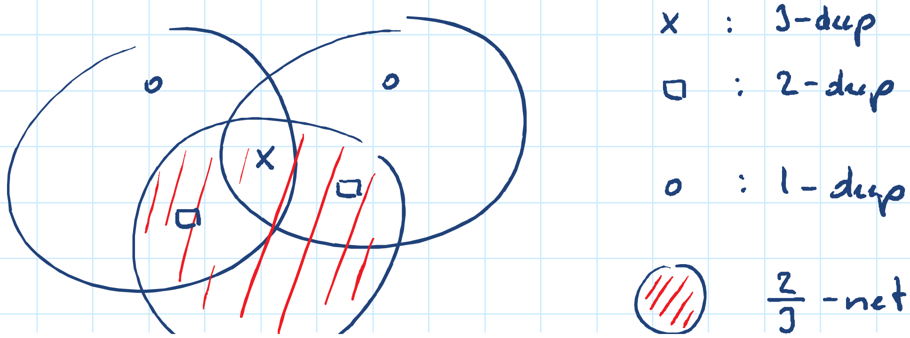
Lot's of work here. Key insight:

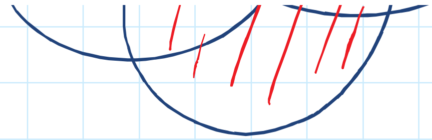
int-gap of canonical set-cover LP linked to existence of small ϵ-nets

Call a point $p \in X$ L-deep if p is in L sets in S.

Let $|S| = N$ and $L \in \mathbb{N}$, $L \leq N$. Then $C \subseteq S$ is an L/N-net if C covers all $\geq L$-deep points in X.

e.g.





 $\frac{2}{3}$-net

[Brönniman & Goodrich '95] Standard Set Cover LP has integrality gap $O(h(\tau^*))$ if there are $1/N$-nets of size $O(N/L \cdot h(N/L))$ for some function h.

$$\tau^* = \min \{ \mathbb{1}^T x \ : \ Ax \geq \mathbb{1} \ , x \geq 0 \ , x \text{ int} \}$$

$\Longrightarrow$ Central idea to approx. geometric problems is to find small ε-nets

[Clarkson '87] [Haussler, Welzl '87]
L/N-nets of size $O(N/L \log N/L)$ exists for many set families (e.g.: triangles, rectangles, disks)

<u>Union Complexity</u> Combinatorial Complexity of the union of n objects in S

Combinatorial complexity $\equiv$ # simple regions in a canonical decomposition of exterior of union of n objects needs to be at most its union complexity $f(n)$ $\Longrightarrow$ [Clarkson, Shor '89]

[Clarkson & Varadarajan '07]
union complexity $O(n\, h(n)) \Longrightarrow$
$\exists\ L/N$-nets of size $O(N/L \cdot h(N/L))$

[Varadarajan '09] improvement to $O(N/L \log h(N/L))$

ex: • a triangle is δ-fat if radius of smallest encl. circle to largest inner circle is $\le \delta$

δ-fat triangles for $\delta = O(1)$ have $O(n \log\log n)$ union bound

$\Rightarrow \frac{L}{N}$ nets of sz $O(\frac{N}{L} \log\log \frac{N}{L})$

(improv to $O(\frac{N}{L} \log\log\log N$ [Va'09]

and further in Ezra et al. '11)

• Axis parallel unit cubes in $\mathbb{R}^3$ have union complexity $O(n)$

$\Rightarrow O(1)$-apx

• n disks in $\mathbb{R}^2$ have union complexity $O(n)$

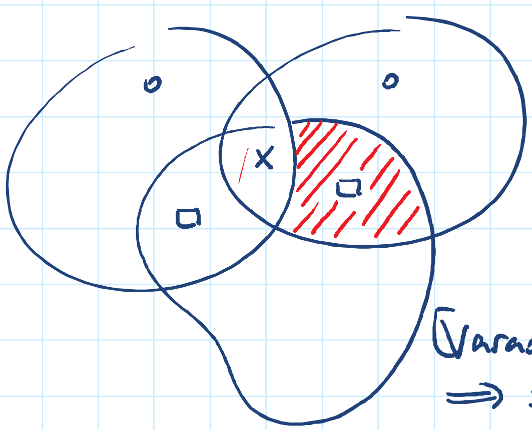
$\Rightarrow O(1)$-apx

[Varadarajan '10] Extend to weighted setting

We use ideas from V10 to prove Thm 2.

Cell Complexity

a cell is a collection of points that are covered by the same sets.



Depth of a cell: # sets covering its points

[Varadarajan '10] low union complexity

$\Rightarrow$ small number of cells of large depth

crucial in his algo

Details Combinatorial view

Let A be 0,1-matrix. Rows A_i, A_j are equivalent if $A_i = A_j$.
Cells of A: equivalence classes.

$$\begin{array}{r} \\ \text{cell 2} \longrightarrow \\ \\ \text{cell 3} \longrightarrow \end{array}\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \text{cell 1} \\ \\ \\ \end{array}$$

Depth of a cell is #1s in its rows.

Def: $f(n,k)$ non-decreasing function in n and k

A $M \times N$ 0,1-matrix

A has shallow cell complexity (SCC) f if $\forall\ 1 \le k \le n \le N$ and for all submatrices A^* of A with n columns the number of cells of depth $\le k$ is $\le f(n,k)$.

An instance of set cover has SCC f iff its set-element inc. matrix does.

Examples (i) general binary matrices have SCC $\binom{n}{k}$

(ii) matrices without $[0,1]$ as submatrix have SCC $k+1$
(0,1 strings of len k without 01 $= k+1$)

Lemma 1 TFC matrices have SCC $O(n)$

Pf: Suppose G has a vertex v of deg ≤ 2.

(I) $\overbrace{\deg_T(v)}^{e} = \overbrace{\deg_{E\setminus T}(v)}^{f} = 1$

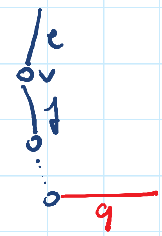
$\Longrightarrow$ v leaf of T and e-row of A has single '1' in f-column

(II) $\overbrace{\deg_T(v) = 1}^{e}, \deg_{E\setminus T}(v) = 0$

$\Longrightarrow$ e-row of A is 0

(III) $\overbrace{\deg_T(v) = 2}^{e,f}$ $\deg_{E\setminus T}(v) = 0$

$\Longrightarrow$ non-tree edge q contains e iff it contains f in fund cycle.



$\Longrightarrow$ rows for e, f have same $0,1$-pattern

Preproc

While G has deg-2 vertex v
$\longrightarrow$ contract one of its inc. edges $e \in T$
(i.e.: delete e-row in A)

effect: decrease # vert of G by 1
delete 0-row, 1-row or duplicate row

Resulting graph $\tilde{G}$ has only deg ≥ 3 nodes.

$$\Longrightarrow \quad 3|V(\tilde{G})| \leq 2|E(\tilde{G})|$$

$$\Longrightarrow \quad (|V(\tilde{G})|-1) \leq 2\big(\overbrace{|E(\tilde{G})| - (|V(\tilde{G})|-1)}^{\text{non-tree edges}}\big) - 3$$

$$= 2n - 3$$

Each iteration of preproc step decreases #vert by 1 and deletes at most one distinct row (cases I & II).

$\Longrightarrow$ A has at most $(2n-3)+(n-1)$
$= 3n-4$ distinct rows. ▨

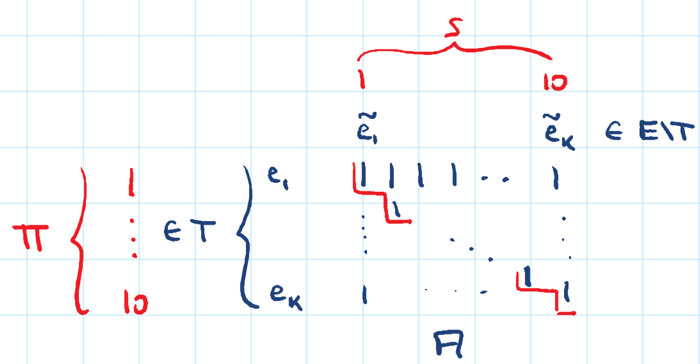
paper says $3n-2$?

Lemma 2 TFC matrix A. $A[s,\pi]$ has SCC $O(nk)$

Pf: Suppose columns of $A[s,\pi]$ are ordered by non-increasing prio s_j.

Effect of adding prio: 0-out a suffix of each row

$\Longrightarrow$ Consider a cell of depth $\le k$ defined by edges $\tilde e_1 \dots \tilde e_k \in E\setminus T$ spanning tree edges $e_1,\dots,e_k$ (intersection of fund cycles of $\tilde e_1,\dots,\tilde e_k$)



$A[s,\pi]_{e_i,\tilde e_j} = 1$ if
$A_{ij} = 1$ & $s_i \geq \pi_j$

cell gives rise to $\le k$ different cells in $A[s,\pi]$ ▨

Main thm $\phi(n)$ non-decr. function, c constant
Let $\mathcal{I}$ be set conn instances of SCC

$$f(n,k) = n\,\phi(n)\,k^{c}$$

$\Rightarrow$ $\exists$ randomized polynomial-time $O(\max\{1, \log \phi(n)\})$-apx for

SC $\min\{c^T x : Ax \geq \mathbb{1}, x \geq 0, x \text{ int}\}$

A: $M \times N$

$\Rightarrow$ CCIP with TFC matrices has $\phi(n) = 1$, $c = 1$ and hence $O(1)$-apx.

Pf of main thm

(1) Compute extreme pt soln x^* to LP relax of SC

Standard x^* has $\leq M$ positive components

Create set multifamily S^* by including $\lfloor 2M x_j^* \rfloor$ copies of each "set" j with $x_j^* \geq \frac{1}{2M}$.

note $\forall i \in [M]$

$$\sum_{\substack{j : x_j^* \geq \frac{1}{2M} \\ A_{ij} = 1}} \underbrace{\lfloor 2M x_j^* \rfloor}_{\geq M x_j^*} \geq M \cdot \sum_{\substack{j : x_j^* \geq \frac{1}{2M} \\ A_{ij} = 1}} x_j^*$$

$$> M\left(1 - M \cdot \frac{1}{2M}\right)$$

$$= M/2$$

Every element $i \in [M]$ is $L := M/2$ deep in S^*.

(2) Find randomized algo that always produces a correct cover of $[M]$ and includes each

(2) Find randomized algo that always produces a correct cover of [M] and includes each set $j \in S^*$ with

Quasi-uniform sampling

$$\text{prob} \le O\left(\frac{\ell(N)}{L}\right)$$

Expected Cover Cost $\le O\left(\frac{\ell(N)}{L}\right) \cdot \sum_{j \in [N]} c_j \lfloor 2M x_j^* \rfloor$

$$\overset{L=M/2}{=} O(\ell(N)) \cdot c^T x^*$$

☑

Missing Piece: Quasi uniform Sampling