Recall (C1P) \( \min \{ c^T x : Fx \geq b, 0 \leq x \leq d, \chi \text{i} \} \)

(CC1P) \( \min \{ c^T x : F_1 x \geq b_1, 0 \leq x \leq d, \chi \text{i} \} \)

Thm 1: CC1P has \( O(n^3 + c) \)-approx if underlying
\( 0,1 \)-C1P and PC1P have int. gap \( \delta \) and \( \epsilon \), respectively.

Remaining Question: Suppose \( \mathcal{P} \) is \( m \times n \) 0,1 network matrix. Can CC1P be approx. well?

Def: Call \( \mathcal{P} \) a **true fundamental cycle (TFC)** matrix if \( \exists (V, E) \) and \( T \subseteq E \) s.t.
\( \mathcal{P} \) has row for each \( e \in T \) and col. for each \( e \in E \setminus T \) and
\[
\mathcal{P}_{e, e^*} = \begin{cases} 
1 & : e \text{ on fund cycle in } T + e^* \\
0 & : \text{ other.}
\end{cases}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & \cdots \\
2 & 4 & 5 & \cdots \\
3 & 5 & 6 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Note: Every \( 0,1 \)-network matrix is a TFC matrix (not converse)

Thm 2 (Chan, Grant, K., Sharpe '12): A TFC matrix \( \Rightarrow \) underlying PC1P has \( O(1) \)
integrality gap

\[ \Rightarrow (1+\epsilon) \text{-apx for } \text{CCIP with F TFC matrix} \]

**Thm 2** is proved using geometric ideas. Canonical problem:

- \( X \) : collection of points in a fixed dim. Euclidean space
- \( S \) : collection of objects - disks, squares, half-spaces...

**Goal:** find \( \min \text{-card} \) (min-card) collection \( C \subseteq S \) that covers all of \( X \)

Let's go back here. Key insight:

- int. gap of canonical set-cover LP linked to existence of small \( \epsilon \)-nets

Call a point \( p \in X \) \( L \)-deep if \( p \) is in \( L \) sets in \( S \).

Let \( |S'| = N \) and \( L \leq N \), \( L \leq N \). Then \( C \subseteq S' \) is an \( L/N \)-net if \( C \) covers all \( \geq L \)-deep points in \( X \).

**E.g.**

\[
\begin{align*}
x &: 1\text{-deep} \\
\circ &: 2\text{-deep} \\
\circ &: 1\text{-deep} \\
\frac{2}{3} &: \text{net}
\end{align*}
\]
Standard Set Cover LP has integrality gap $O(h(T^*) \log N)$ if there are $\frac{1}{N}$-nets of size $O(N \cdot h(N))$ for some function $h$.

$T^* = \min \{ \Delta(x) : \forall x \geq 1, x \geq 0, x \in \mathbb{Z} \}$

Central idea: Approx. geometric problem is to find small $\epsilon$-nets.

[Clarkson '87, Hanuss, Uetz '87]

$\frac{1}{N}$-nets of size $O(N \log N)$ exist for many set families (e.g., triangles, rectangles, disks).

Union complexity

\[ \text{Combinatorial complexity of the union of n objects in S} \]

Combinatorial complexity = \# simple regions in a canonical decomposition of exterior of union of n objects, needs to be at most its union complexity $f(n) \Rightarrow$ [Clarkson, Shar '89]

[Clarkson & Varadarajan '07]

Union complexity $O(n h(n)) \Rightarrow$

$\exists \frac{1}{N}$-nets of size $O(N \log N \cdot h(N))$

[Varadarajan '09] Improvement to $O(N \log^2 h(N))$
**Example:** A triangle is $S$-jal if radius of smallest enclosing circle to largest inscribed circle is $\leq S$.

$S$-jal triangles for $S = O(1)$ have $O(n \log \log n)$ unit-diamonds.

$$\Rightarrow \frac{1}{N} \text{ # of } x \in \mathbb{R} \quad O \left( \frac{N}{n} \log \log \frac{N}{n} \right)$$

(Improved to $O \left( \frac{N}{n} \log \log \log N \right)$ [Va'09]

and further in Efra et al. '11)

- Axis-parallel unit cubes in $\mathbb{R}^3$
  have unit-complexity $O(n)$
  $$\Rightarrow O(1) - \text{apx}$$

- $n$ disks in $\mathbb{R}^2$ have unit-complexity $O(n)$
  $$\Rightarrow O(1) - \text{apx}$$

[Varadarajan '10] Extend to weighted setting

We use ideas from [10] to prove Thm 2.

**Cell Complexity:** A cell is a collection of points that are covered by the same sets.

- Depth of a cell: # sets covering its points

[Varadarajan '10] low unit-complexity $\Rightarrow$ small number of cells of large depth crucial in his algo
Details  Combinatorial view

Let $A$ be a $0,1$-matrix. Rows $R_i, R_j$ are equivalent if $R_i = R_j$.
Cells of $A$ : equivalence classes.

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{pmatrix}
$$

Depth of a cell is $\#1$s in its row.

**Def:** \( f(n,k) \) non-decreasing function in \( n \) and \( k \)
A \( M \times N \) $0,1$-matrix

$A$ has **shallow cell complexity (SCC)** \( f \)
if $\forall 1 \leq k \leq n \leq N$ and
for all submatrices $A'$ of $A$ with $n$ columns
the number of cells of depth $\leq k$ is \( \leq f(n,k) \).

An instance of set cover has SCC \( f \) if
it is set-element incidence matrix does.

**Examples**

(i) general binary matrices have SCC \( \binom{n}{k} \)

(ii) matrices with $[0,1]$ as submatrix
have SCC $k+1$
(0,1 strings of km k without 01 = $k+1$)

**Lemma** TFC matrices have SCC $O(n)$
Pf: Suppose \( G \) has a vertex \( v \) of deg \( \leq 2 \).

1. \( \deg_T(v) = \deg_{\text{INT}}(v) = 1 \)
   \[ \Rightarrow v \text{ is a leaf of } T \text{ and } e - \text{root of } T \text{ has a single } '1' \text{ in } \frac{1}{2} - \text{column} \]

2. \( \deg_T(v) = 1, \deg_{\text{OUT}}(v) = 0 \)
   \[ \Rightarrow e \text{ - root of } T \text{ is } 0 \]

3. \( \deg_T(v) = 2, \deg_{\text{OUT}}(v) = 0 \)
   \[ \Rightarrow \text{ non-tree edge } q \text{ contains } e \text{ if it contains } \frac{1}{2} \text{ and } q \text{ contains } 1 \text{ - find such } q \]
   \[ \Rightarrow \text{ now for } e, q \text{ have same } q_1 - \text{path} \]

**Proc**

While \( G \) has deg-2 vertex \( v \)
- contract one of its inc. edges \( e \in T \)
  (i.e.: delete \( e - \text{root in } T \))

**Effect:**
- decrease # of vertices \( G \) by 1
- delete \( 0 - \text{root, } 1 - \text{root or duplicate root} \)

Resulting graph \( \widetilde{G} \) has only deg \( \geq 3 \) nodes.

\[ 3|V(\widetilde{G})| \leq 2|E(\widetilde{G})| \text{ non-tree edges} \]

\[ (|V(\widetilde{G})| - 1) \leq 2(|E(\widetilde{G})| - |V(\widetilde{G})| - 1) - 3 \]
\[ = 2n - 3 \]
Each iteration of preproc step decreases \#vot by 1 and deletes at most one distinct row (case \( k = 1 \)).

\[ \Rightarrow \text{\# of rows } \leq (2n-1) + (n-1) \]
\[ = 3n - 2 \text{ distinct rows.} \]

\[ \text{papr says } 3n - 2 \text{ ?} \]

\textbf{Lemma 2} \quad \text{TFC matrix } \mathbf{F}, \mathbf{R}_{\mathbf{C}(s, \pi \bar{t})} \text{ has SCG } O(nk) \]

\textbf{PA:} \quad \text{Suppose columns of } \mathbf{R}_{\mathbf{C}(s, \pi \bar{t})} \text{ are added by non-increasing prior } s_j.

\textbf{Effect of adding prior:} \quad 0 - out a suffix of each row

\[ \Rightarrow \text{Consider a cell of depth } \leq k \text{ defined by edges } \tilde{e}_1, \ldots, \tilde{e}_k \in \text{ET spanning tree} \]
\[ \text{edges } e_1, \ldots, e_k \text{ (intersection of fund cycles } A, \Delta \tilde{e}_1, \ldots, \tilde{e}_k) \]

\[ \Phi_{\mathbf{C}(s, \pi \bar{t})} e_i, \tilde{e}_j = 1 \quad \text{if} \]

\[ \mathbf{F}_{i,j} = 1 \quad \& \quad s_i > s_j \]

Cell gives rise to \( \leq k \) different cells in \( R_{\mathbf{C}(s, \pi \bar{t})} \)

\textbf{Main Thm} \quad \phi(n) \text{ non-dec. function, C constant.}

Let \( \mathcal{X}_G \) be set contain instances of SCG.
\[ f(n, k) = n \cdot \phi(n) \cdot k^v \]

\[ \exists \text{ randomized polynomial-time } \]

\[ O(\max \{1, \log \phi(n)\}) - \text{approx.} \]

\[ \text{min } \{ c \in \mathbb{R}^n : H \cdot x \geq 2 \cdot 1, x \geq 0, x \text{ int}\} \]

\[ \uparrow \text{ max } \]

\[ \Rightarrow \text{ CCP with TFC method has } \phi(n) = 1, c = 1 \]

\[ \text{and hence } O(1) - \text{approx.} \]

---

**Proof of Main Theorem**

1. Compute extreme pt sol - \( x^* \) to LP relax of (5)

   **Standard** \( x^* \) has \( \leq M \) positive components

   Create set multi-family \( S^* \)

   by including \( \lfloor \frac{2M}{M} x^*_j \rfloor \) copies of each "set" \( j \) with \( x^*_j \geq \frac{1}{2M} \).

   **Note** \( \forall i \in [M] \)

   \[ \sum_{j : x^*_j \geq \frac{1}{2M}} \lfloor \frac{2M}{M} x^*_j \rfloor \geq M \cdot \sum_{j : x^*_j \geq \frac{1}{2M}} x^*_j \]

   \[ \sum_{j : x^*_j \geq \frac{1}{2M}; H^T u = 1} \geq M x^*_j \]

   \[ > M (1 - M \cdot \frac{1}{2M}) \]

   \[ = \frac{M}{2} \]

   Every element \( i \in [M] \) is \( L_i = M/2 \) deep in \( S^* \).

2. Find randomized algo that always produces a correct conv of \( [M] \) and includes each
Find randomized algo that always produces a correct conv of \([M]\) and includes each set \(j \in S^*\) with
\[
\Pr[d] = O\left(\frac{e(N)}{L}\right)
\]

Expected gro cost
\[
\leq O\left(\frac{e(N)}{L}\right) \cdot \sum_{j \in [M]} c_j L^2 M x_j^* \]

\[
L = M^2
\]
\[
= O(e(N)) \cdot c^T x^*
\]

**Missing Piece:** Quasi Uniform Sampling