CR-Cover (2)

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Recall (CIP) $\min \{c^T x : Ax \geq b,\ 0 \leq x \leq d,\ x \text{ int}\}$

A: $m \times n$ 0,1

Want to solve $s \in \mathbb{N}^n$

(CCIP) $\min\ c^T x$

s.t. $A[s]\, x \geq b$ ← $\mathbb{N}^m$

$0 \leq x \leq d$ ← $\mathbb{N}^n$

$A[s]_{ij} = A_{ij} \cdot s_j$

Q: How well can we approximate CCIP if CIP can be approximated well?

note: CCIP can be harder than underlying CIP

$\min\ c^T x$
s.t. $\mathbb{1}^T x \geq b$
$0 \leq x \leq \mathbb{1}$

is easy, but CCIP is equiv. to min Knapsack

underlying CIP

$\min\ c^T x$
s.t. $Ax \geq b$
$0 \leq x \leq d$
x int

underlying PCIP

$\min\ c^T x$
s.t. $A[s,\pi]\, x \geq \mathbb{1}$
$0 \leq x \leq d$
x int

$A[s,\pi]_{ij} = \begin{cases} 1: & A_{ij} = 1 \ \& \ s_j \geq \pi_i \\ 0: & \text{othw.} \end{cases}$

Thm CCIP has $O(\gamma + \beta)$-apx if underlying CIP and PCIP have int-gap γ, β.

Canonical LP for CCIP is bad

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & A[s]\,x \geq b \\ & 0 \leq x \leq d \end{aligned}$$

Knapsack instance:
Items $[n]$, each $j \in [n]$
has size $s_j \in \mathbb{N}$
and cost $c_j \in \mathbb{N}$

EX

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & s^T x \geq n \\ & 0 \leq x \leq \mathbb{1} \end{aligned}$$

$c_1, \dots, c_{n-1} = 0 \quad c_n = 1$

$s_1, \dots, s_{n-1} = 1 \quad s_n = n$

$opt_{IP} = 1 \quad opt_{LP} = \frac{1}{n}$

Strengthen canonical LP

Replace
$$\sum_{j=1}^{n} A[s]_{ij}\, x_j \geq b_i \quad \forall i \in [m]$$

by Knapsack Cover inequality

$$\sum_{j=1}^{n} A^F[s]_{ij}\, x_j \geq b_i^F \quad \forall F \subseteq [n],\ i \in [m]$$

$$0 \leq x \leq d$$

Here:
$$b_i^F = \max\{0,\ b_i - \overbrace{\sum_{j \in F} A[s]_{ij}\, d_j}^{\text{max row } i \text{ contrib of } F}\}$$

$$A^F[s]_{ij} = \begin{cases} \min\{A[s]_{ij},\ b_i^F\} & : j \notin F \\ 0 & : j \in F \end{cases}$$

in EX $\quad F = \{1, \dots, n-1\} \quad b^F = n - s(F) = 1$

$$s_n^F = \min\{n, b^F\} = 1$$

$$\Rightarrow \quad x_n \geq 1$$

<u>Bummer</u> don't know how to solve strengthened LP

Call $x^* \in \mathbb{R}^n_+$ an α-relaxed soln to strengthened LP ($\alpha \in [0,1]$) if $c^T x^* \leq \underbrace{opt_P}_{\text{opt strengthened LP val}}$ and

$$\sum_{j \in [n]} A^F[s]_{ij} \, x_j^* \geq b_i^F \qquad \forall i \in [m]$$

and $$F = \{ j \in [n] : x_j^* \geq \alpha \cdot d_j \} \qquad \circledast$$

Thm [Carr, Fleischer, Leung, Phillips '00]
α-relaxed soln's to strengthened LP can be computed efficiently.

<u>Pf-Sketch</u> use the Ellipsoid method.
Start with original CCIP relaxation without Knapsack cover LP.

For candidate pt x^* check Knapsack cover ineq for F as in $\circledast$.
Sat $\xrightarrow{y}$ terminate
$\Downarrow n$
add cut for F. This gives relaxed feasibility.
Use binary search to optimize. $\blacksquare$

<u>So:</u> Let x^* be an $\underset{1/24}{\alpha}$-relaxed soln to streng. LP

$$F = \{ j : x_j^* \geq \alpha d_j \}$$

Supp x^{int} feasible for residual prob:

(Res) min $c^T x$

s.t.

$$A^F[s]\, x \geq b^F$$
$$0 \leq x \leq d$$

Note: Let $\bar{x}_j = \begin{cases} x^*_j & : j \notin F \\ 0 & : \text{othw.} \end{cases} \quad \forall j \in [n]$

$\Longrightarrow$ $\bar{x}$ feas for (Res)

Lemma 1 Let x^* be α-relaxed soln for strengthened CCIP LP. Let x^{int} be int feas for (RES) s.t. $c^T x^{int} \leq \beta\, c^T \bar{x}$ and let

$$z_j = \begin{cases} d_j & : j \in F \\ x^{int}_j & : j \notin F \end{cases} \quad \forall j \in [n]$$

Then z is feasible for orig CCIP instance and $O(\frac{1}{\alpha} + \beta)$ - apx.

Pf: easy from def of (Res) and α-relax $\square$

Todo Approximate residual prob $\Rightarrow$ use small integrality gap of underlying 0,1 - CIP/PCIP

Technical prelude: may assume that b_i^F and s_j are powers of 2 $\forall i, j$

$\bar{b}_i$: smallest power of 2 $\geq b_i^F$ $\forall i$

$\bar{s}_j$: largest power of 2 $\leq s_j$ $\forall j$

$\bar{s}_j$: largest power of 2 $\le s_j \ \forall j$

Lemma 2 $\boxed{y = 4\bar{x}}$ is feasible for

(Res^2)

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & A^F[\bar{s}]x \ge \bar{d} \\ & 0 \le x \le 4\alpha d \end{aligned}$$

Pf: easy

Plan Consider row $i \in [m]$ and partition cols:

[i-large cols] $\mathcal{L} = \{ j \in [n] : A_{ij} = 1,\ \bar{s}_j \ge \bar{d}_i \}$

[i-small cols] $\mathcal{S} = \{ j \in [n] : A_{ij} = 1,\ \bar{s}_j < \bar{d}_i \}$

Call a row $i \in [m]$ large if

$$\sum_{j \in \mathcal{S}_i} A^F[\bar{s}]_{ij}\, y_j \le \sum_{j \in \mathcal{L}_i} A^F[\bar{s}]_{ij}\, y_j$$

Call row i small otherw.

Then Find pair $x^{int,\mathcal{L}}$ and $x^{int,\mathcal{S}}$ of integral solutions for large & small rows respectively.

$$x^{int}_j = \max\{ x^{int,\mathcal{L}}_j ,\ x^{int,\mathcal{S}}_j \}$$

$\Longrightarrow x^{int}$ is final feas. soln

(I) Small rows: Computing $x^{int,\mathcal{S}}$

Let i be a small row and hence

$$(*) \quad 2 \cdot \sum_{j \in S_i} A^F[\bar{s}]_{ij} y_j \geq \bar{b}_i$$

$$\bar{s}_{max} := \max_{j \in [n]} \bar{s}_j \qquad \bar{s}^{(t)} = \frac{\bar{s}_{max}}{2^t}$$

Let $\mathcal{C}^{(t)} = \{ j \in [n] : \bar{s}_j = \bar{s}^{(t)} \}$
be the set of columns
with supply $\bar{s}^{(t)}$.

note: let $t_i = \log\left(\frac{s_{max}}{\bar{b}_i}\right) + 1$

i-small columns are in $\bigcup_{t \geq t_i} \mathcal{C}^{(t)}$

$$\bar{s}_j = \frac{s_{max}}{2^t} < \bar{b}_i \Rightarrow t > \underbrace{\log \frac{s_{max}}{\bar{b}_i}}_{\text{int}}$$

$$\Rightarrow t \geq \left(\log \frac{s_{max}}{\bar{b}_i}\right) + 1$$

Then define the class-t supply in row i as

$$\bar{b}_i^{(t)} = \begin{cases} 2 \cdot \sum_{j \in \mathcal{C}^{(t)}} A^F[\bar{s}]_{ij} y_j & : t \geq t_i \\ 0 & : t < t_i \end{cases}$$

$$(*) \Rightarrow \quad \sum_{t \geq 0} \bar{b}_i^{(t)} \geq \bar{b}_i$$

Grouping Focus on group $\mathcal{C}^{(t)}$ of colums for $t \geq 0$. Then

see below

$$6 \cdot \sum_{j \in \mathcal{C}^{(t)}} A_{ij} y_j \geq \left\lfloor 3\bar{\delta}_i^{(t)} / \bar{s}^{(t)} \right\rfloor$$

Clearly true if $t < t_i$ as then $\bar{\delta}_i^{(t)} = 0$.

So suppose $t \geq t_i$. By def:

$$2 \cdot \sum_{j \in \mathcal{C}^{(t)}} A_{ij} [\bar{s}] y_j = \delta_i^{(t)}$$

$$\Longleftrightarrow 6 \sum_{j \in \mathcal{C}^{(t)}} A_{ij} y_j = \frac{3\bar{\delta}_i(t)}{\bar{s}^{(t)}}$$

Consider

(P_t)

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & \sum_{j \in \mathcal{C}^{(t)}} A_{ij} x_j \geq \left\lfloor \frac{3\bar{\delta}_i^{(t)}}{\bar{s}^{(t)}} \right\rfloor \quad \forall \text{ small } i \quad (i) \\ & 0 \leq x \leq d \quad (ii) \end{aligned}$$

and note that $6y = 24 \cdot \bar{x}$ satisfies (i) by above argument.
Also since $\bar{x}_j \leq d_j/24$, $6y$ satisfies (ii) as well.

Assumption $\Longrightarrow$ $\exists$ integral soln $\hat{x}^t$ sat (i) and (ii) and

$$\sum_{j \in \mathcal{C}^{(t)}} c_j \hat{x}^t \leq 6 \cdot \gamma \cdot \sum_{j \in \mathcal{C}^{(t)}} c_j y_j$$

$$= 24\gamma \cdot \sum_{j \in \mathcal{C}^{(t)}} c_j \bar{x}_j$$

Assemble solution for small rows:

$$x_j^{int, S} = \hat{x}_j^t \quad \text{if } j \in \mathcal{C}^{(t)}$$

Lemma 3 $x^{int, S}$ has cost $\leq 24\delta\, c^T \bar{x}$
and for each small row i:

$$\sum_{t \geq t_i} \sum_{j \in \mathcal{C}^{(t)}} A_{ij}\, \bar{s}^{(t)} x_j^{int, S} \geq \bar{b}_i$$

Pf: Cost part is immediate. Feasibility inherently uses scaling factor 3 introduced in grouping step above. ($\rightarrow$ rhs of (i) in P_t)

Rather technical and not interesting step
$\Rightarrow$ see paper ▨

(II) Large Rows

recall: $i \in [m]$ is large if

$$\bar{b}_i \leq 2 \cdot \sum_{j \in \mathcal{L}_i} A^F[\bar{s}]_{ij}\, y_j$$

$$\bar{s}_j \geq \bar{b}_i \Rightarrow A^F[\bar{s}]_{ij} = \bar{b}_i$$

$$\Longleftrightarrow \quad 2 \cdot \sum_{j \in \mathcal{L}_i} A_{ij} \cdot y_j \geq 1$$

So: $2y$ is feasible for

(L) $\min c^T x$
s.t.

$$\underbrace{\sum_{j \in \mathcal{L}_i} A_{ij}\, x_j \geq 1}_{\text{equiv: } \sum A[\bar{s}, \bar{b}]_{ij}\, x_j \geq 1} \quad \forall \text{ large } i \in [m]$$

$$x \geq \mathbb{0}$$

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$x \geq \mathbb{1}$ equiv: $\sum_{j \in [n]} A[\bar{S}, \bar{\delta}]_{ij} x_j \geq 1$

$\Uparrow$

This is a priority covering prob.

By assumption: $\exists$ integral soln $x^{int, 2}$ s.t.

$$c^T x^{int, 2} \leq 2\omega\, c^T y \overset{y = 4\bar{x}}{=} 8\omega\, c^T \bar{x}$$

$\uparrow$

$2y$ feasible for relax $\blacksquare$

Complete pf: $x^{int} = \max\{x^{int, S}, x^{int, 2}\}$
is feasible and $O(\omega + \delta)$-apx for
Res?