Flow Cover - CS08

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Minimum Knapsack   items F

For each item $i \in F$, have value $u_i \geq 0$ and weight $f_i > 0$.

Demand $D > 0$.

Goal: Choose min-weight collection $S \subseteq F$ s.t. $u(S) \geq D$.

Natural LP

(P) $\min \sum_i f_i y_i$

s.t. $\sum_i u_i y_i \geq D$

$\cancel{y_i \in \{0,1\}}$ $\forall i$

$0 \leq y \leq 1$

LP is not good

bad example with $F = \{1,2\}$

$u_1 = D-1 \qquad u_2 = D$

$f_1 = 0 \qquad f_2 = 1$

$opt^{IP} = 1$

but: $y_1 = 1, y_2 = \frac{1}{D}$ feasible for (P) and $val = \frac{1}{D}$

$\Longrightarrow$ integrality gap is $\geq D$

Strengthen (P)

Problem above: object 1 alone is not feasible even if object 1 is chosen, all of obj. 2 needs to be picked as well.

more generally let $A \subseteq F$ s.t. $u(A) < D$ then any feasible soln must pick $S \subseteq F \setminus A$ s.t.

$$(*) \quad \sum_{i \in S} u_i \geq D(A) = D - u(A)$$

Inequality $(*)$ must hold even if we replace u_i by

$$u_i(A) = \min \{u_i, D(A)\}$$

$$\Longrightarrow \sum_{i \in F \setminus A} u_i(A) y_i \geq D(A) \quad \forall A \subseteq F$$

is a valid constraint for IP.

(P_2) $\quad \min \; f^T y$

$$\text{s.t.} \quad \sum_{i \in F \setminus A} u_i(A) y_i \geq D(A) \quad \forall A \subseteq F \qquad [v_A]$$

$$y \geq 0$$

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note: we don't need upper bounds any longer

(*) is called a Knapsack cover (or flow cover) inequality. ($\hookrightarrow$ [Aardal, Pochet, Wolsey '95])

Dual of (P₂)

$$(D_2) \quad \max \sum_{A \subseteq F} D(A)\, v_A$$

$$\text{s.t.} \quad \sum_{i \in F \setminus A} u_i(A)\, v_A \leq f_i \quad \forall i \in F \quad (**)$$

$$v \geq 0$$

Primal-Dual Algo

(1) $v = 0$ dual feasible
$A = \emptyset$

(2) while A is infeasible

(3) increase $v(A)$ as much as possible while maintaining dual feasibility
$\rightarrow$ (**) for some $i \notin A$ is now tight

does not affect dual constr. of $i \in A$

(4) Add i to A

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Thm Let (A^*, v^*) be the pair of primal and dual feasible solutions computed by above algo.

$$\Longrightarrow \quad f(A^*) \le 2 \cdot \sum_A D(A)\, v_A \le 2 \cdot opt_{P_2}$$

Pf: Suppose that $e \in A^*$ was added last.

$$\Longrightarrow \quad D(A^* - e) > 0 \quad \text{as } A^* - e \text{ infeasible}$$

also $v_A > 0$ only if $A \subseteq A^* - e$

Let $i \in A^* \setminus e$ be any other obj. in the solution and $A \subseteq A^* \setminus \{i, e\}$ with $v_A^* > 0$:

$$\Longrightarrow \quad u_i(A) = u_i$$

as adding i did not read feasibility.

$$\sum_{i \in A^*} f_i = \sum_{i \in A^*} \overbrace{\sum_{A : i \notin A} u_i(A)\, v_A^*}^{= f_i}$$

$$= \sum_A v_A^* \sum_{i \in A^* \setminus A} u_i(A) \quad \leftarrow$$

v_A^* contr. to all obj. in A^* that are not in A

$$= \sum_A v_A^* \big(\underbrace{u(A^* - e) - u(A)} + u_e(A) \big)$$

no capping of elements in $A^* \setminus A$ except c

$$< \sum_{A} v^*_A \left(\underbrace{D - u(A)}_{D(A)} + \underbrace{u_c(A)}_{\le D(A)} \right)$$

$$\le 2 \sum_{A} D(A)\, v^*_A \qquad \blacksquare$$