Steiner Trees: bidirected cut

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Have seen: hypergraphic LPs for Steiner trees and $(\ln 4+\epsilon)$-apx based on iteratively rounding these [Byrka et al. '11]

also: [Goemans et al. '12] showed that these LPs have int gap $\leq \ln 4$

Algorithms are elegant but need to solve large LPs!

Recall bidirected cut relaxation

Instance: $G=(V,E)$, $R \subseteq V$, $c: E \to \mathbb{R}_+$

Create digraph D on V by inserting (u,v), (v,u) of cost c_{uv} $\forall$ $uv \in E$.

Pick $r \in R$.

$$(B_r) \qquad \min \underbrace{\sum_a c_a x_a^{(r)}}_{c^T x^{(r)}} \qquad =: opt_B$$

$$\text{s.t.} \quad x^{(r)}(\delta^+(S)) \geq 1 \quad \forall S \subseteq V \setminus r$$

$$x^{(r)} \geq 0$$

Observations ① optimum value of (B_r) is independent of r: let $x^{(r)}$ be an opt. soln to (B_r) and choose $r' \in R \setminus r$.

Feasibility $\Rightarrow$ $\exists$ unit-value r', r - flow f in D with capacities $x^{(r)}$

Obtain $x^{(r')}$ from $x^{(r)}$ by reversing f:

$$x^{(r')}_{uv} = \begin{cases} x^{(r)}_{uv} - f_{uv} & : f_{uv} \geq 0 \\ x^{(r)}_{uv} + f_{vu} & : f_{vu} > 0 \end{cases}$$

Claim: $x^{(r')}$ is feasible for $B_{r'}$ and $c^T x^{(r)} = c^T x^{(r')}$

Pf: ex.

($\rightarrow$ [Goemans & Myung '93])

2 $\quad opt_B \geq opt_D^{lp}/2$

Recall that $\quad opt_U = \min \; c^T x$

$$\text{(U)} \qquad \text{s.t.} \quad x(\delta(S)) \geq 1 \quad \forall S \subseteq V: S \cap R \neq \emptyset, R \setminus S \neq \emptyset$$
$$x \geq 0$$

and $opt_U \geq opt_U^{lp}/2 = opt_D^{lp}/2$.

So suffices to show that $opt_B \geq opt_U$.

Let $x^{(r)}$ feasible for B_r and "project" it onto E: $\forall uv \in E$

$$x_{uv} = x^{(r)}_{uv} + x^{(r)}_{vu}$$

$\Longrightarrow$ x is feasible for (U) and $c^T x = c^T x^{(r)}$.

Open: Is the integrality gap of B smaller than 2?

True if G is quasi-bipartite: $u, v \in V \setminus R$
$\Longrightarrow uv \notin E$

[Chakrabarty, K., Pritchard '10]

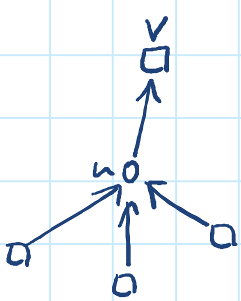
Why care? QB instances show up in NP-hardness reductions for Steiner tree.

more general: [Feldmann, K., Olver, Sanità '14]

Strategy: Compute optimal soln to (B) $\longrightarrow x$

natural decomposition strategy:

Suppose that $\exists u \in V \setminus R$ and $(u,v) \in \mathrm{supp}(x)$.
Consider the min x-value on arcs $\underbrace{\{(u,v)\} \cup (\delta^-(u) \cap \mathrm{supp}(x))}_{(*)} \rightarrow \epsilon > 0$



Transfer ϵ from x on arcs (*) onto full comp. corresp to (*). Continue.

Two simplifying assumptions:
(i) no two edges in G have same cost and (ii) no edges between terminals.

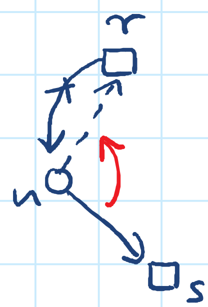
Thm x optimum soln to BCR. The natural strategy above yields feasible soln for DCR.

Pf: In the following let $x^{(r)}$ be the "r-rooted" solution to B_r obtained from x.

Consider $u \in V \setminus R$ and $N(u) = \{v \in R : uv \in E\}$ be its neighbors.

Lemma 1: Let $r \in N(u)$. Then $x^{(r)}_{ru} = x^{(r)}_{us} = 0$ $\forall s \in N(u)$ with $c_{us} > c_{ur}$.

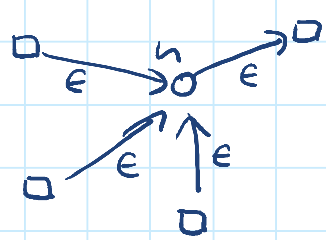
Pf.



Capacity on (r,u) can be removed and cap on (u,s) can be shifted to (u,r) without affecting feasibility or increasing cost. ∎

We prove thm in 2 steps.

(I) Create a solution to D in auxiliary digraph D s.t. (a) the arcs incident to each $u \in V \setminus R$ form a directed full component and (b) the x-val on each arc incident to u is the same



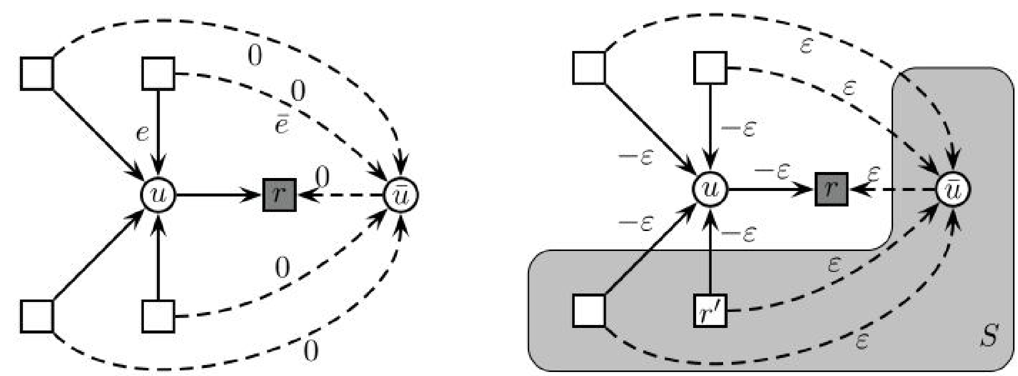
(II) the induced DCR solution (one oriented DC for each $u \in V \setminus R$) is feasible.

Solving I Let $u \in V \setminus R$ and

$$r = \operatorname{argmin} \{ c_{ur} : r \in N(u) \}$$

$$H = \{(u,r)\} \cup \{ (s,u) : x^{(r)}_{su} > 0 \}$$

Add a copy $\bar{u}$ of u to D as well as a copy of H.



(a) capacities $x^{(r)}$ (b) capacities $\bar{x}^{(r)}$

Choose $0 < \varepsilon \le \min_{e \in H} x^{(r)}_e$ small enough.

Let $\bar{e}$ be the natural copy of e just created. Obtain $\bar{x}^{(r)}$ from $x^{(r)}$ by transferring ε from $x^{(r)}_e$ to $\bar{x}^{(r)}_{\bar{e}}$ $\forall e \in H$.

Claim $\bar{x}^{(r)}$ is feasible for B_r in the aux digraph for some $\epsilon > 0$.

Pf: Suppose not and hence $\exists\, S \subseteq V \setminus r$:

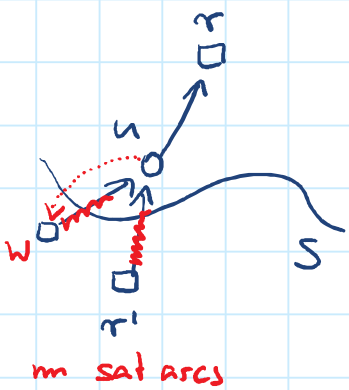
$$\bar{x}^{(r)}(\delta^+(S)) < 1$$

Must have $x^{(r)}(\delta^+(S)) = 1$ as othw. could choose ϵ smaller.

Also: $u \notin S$ as othw. $|\delta^+(S) \cap H| \le 1$ and transfer would not decrease

$$x^{(r)}(\delta^+(S))$$

$$\Longrightarrow \; |\delta^+(S) \cap \delta^-(u)| \ge 2$$



$$r' = \underset{v \in N(u) \cap S}{\operatorname{argmin}} \; c_{uv}$$

Lemma 1 $\Longrightarrow$ $x_e^{(r)} = 0 \;\; \forall e \in (\delta^-(u) \cup \delta^+(u)) \setminus H$

Since $x^{(r)}(\delta^+(S)) = 1$, unit flow f from r' to r saturates arcs in $\delta^+(S) \cap \delta^-(u)$.

$\Longrightarrow$ relocating the root from r to r' reverses the arcs in $\delta^+(S) \cap \delta^-(u)$

Let $(w,u) \in \delta^+(S) \cap \delta^-(u)$, $w \neq r$.

$$\Longrightarrow \quad x_{uw}^{(r')} > 0 \text{ and } c_{uw} > c_{ur'} \quad ⚡ \quad \blacksquare$$

Apply the transfer strat. above repeatedly until we arrive at soln x^* to B_{r^*} in some aux graph and $r^* \in R$ s.t. support arcs around each $u \in V \setminus R$ correspond to directed fc. (k_u, r_u) and all arcs carry flow ϵ_u. Let

$$y_{k_u r_u} = \epsilon_u$$

ex

$\forall u \in V \setminus R$. Then y is feasible for DCR. $\blacksquare$