Let $S^* = S^1$ be a Steiner tree

Strategy so far: In each iteration $i \geq 1$, solve DCR $\rightarrow \bar{x}$
sample full component K^i from $\bar{x}$-induced dist
Remove certain edges from $S^i \rightarrow S^{i+1}$

$\boxed{\text{invariant:}}$ $K^1 \cup \dots \cup K^{i-1} \cup S^i$ spans R

Progress measure: contractions in each iteration yield $(1 - 1/m)$-factor decrease in exp. cost of some terminal spt.

main idea: can delete $\text{Drop}_{S^i}(K^i)$ from S^i in each step i

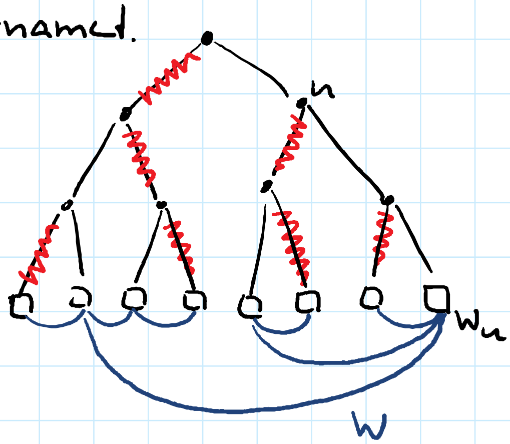
parts of term.spt

Now: pick edges to delete more uniformly
but: maintain invariant

Witness Tree: S^* wlog assume that it is binary and all terminals are at leaves

Think: S^* is a tournament.
Each internal node is a match.

For each such match pick a random loser. All losers: L.



Note: The winner w_u at internal node u of S^* has won all his games up to u.

$\Longrightarrow \exists\, u, w_u$ - path in $S^* \setminus L$

for all internal nodes u.

Let W be all pairs of players that play against each other in the tournament at some point:

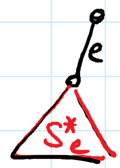
path between u,v in S^*

$$W = \{ uv \in R \times R : |S^*_{uv} \cap L| = 1 \}$$

Immediate: W is a spanning tree on R

subtree of S^* hanging off e

For each $e \in S^*$ let $W(e)$ be the matches the winner of S^*_e undertakes (outside S^*_e):



$$W(e) = \{ uv \in W : e \in S^*_{uv} \}$$

Observations

(A) If $e \in L \Rightarrow$ winner of S^*_e loses immediately
$\Rightarrow |W(e)| = 1$

(B) $\Pr(|W(e)| = q) \le \frac{1}{2^q}$ ($\Leftarrow S^*_e$ winner needs q wins, each with prob $1/2$)

Algo: In each iteration $i \geq 1$ solve DCR for instance induced by $R^i \subseteq R \Rightarrow \bar{x}^i$
Sample K^i from $\bar{x}^i$ and $R^{i+1} = R^i / K^i$

For analysis only: Let

note: old refers to orig W

For analysis only: Let

$$D_W(K^i) = \{ D \subseteq W : |D| = |K^i| - 1, \; W \setminus D \cup K^i \text{ connects } V(W) \}$$

possible drop sets. Pick $\bar{D}_W(K^i) \in D_W(K^i)$ according to distribution p^{K^i} to be chosen. Now mark all edges in $\bar{D}_W(K^i)$.

Delete e from S^i if all edges in $W(e)$ are marked $\to S^{i+1}$.

More observations

Let $W' \subseteq W$ be unmarked edges at beg. of iteration i. Note that

$$W \setminus W' \subseteq \bar{D}_W(K^1) \cup \ldots \cup \bar{D}_W(K^{i-1})$$

$\Rightarrow$ $W' \cup K^1 \cup \cdots \cup K^{i-1}$ connects R

Also: $uv \in W' \Rightarrow uv \in W(e) \; \forall e \in S^*_{uv}$ and hence $S^*_{uv} \subseteq S^i$

$\Longrightarrow$ $S^i \cup K^1 \cup \ldots \cup K^{i-1}$ connects R

So: as soon as every edge of W is marked $K^1 \cup \ldots \cup K^i$ is a Steiner tree

Lemma Can find distributions p^K on $D_W(K)$ $\forall K \in \text{supp}(\bar{x})$, $\bar{x}$ feas DCR soln, s.t. $\Pr[f \text{ marked}] \geq \frac{1}{M}$ $\forall f \in W$.

What does this do?

Pick $e \in S^*$ with $|U(e)| = q$. Each element in $U(e)$ is marked with prob $\geq 1/M$.

think: Coupons collector $\Rightarrow$ takes expected $\frac{M}{q}$ rounds to mark 1st elem in $U(e)$

Intuition not quite correct...

It then takes $\frac{M}{q-1}$ rounds to mark 2nd ...

After an expected $\sum_{i=1}^{q} \frac{M}{i} = M H_q$ rounds $U(e)$ is fully marked on expectation.

$c(S^i)$ is an upperbound on $c^T \bar{x}^i$

$\Rightarrow$ on expectation, each $e \in S^*$ contributes to $M H_{|U(e)|}$ of these bounds

PF:

- j is marked if K is picked and $D \in \mathcal{D}_W(K)$ chosen s.t. $j \in D$

$\Rightarrow$ want

$$\sum_{\substack{(K,D):\, D \in \mathcal{D}_W(K) \\ j \in D}} \frac{\bar{x}_K}{\cancel{M}} \cdot p_D^K \geq \frac{1}{\cancel{M}}$$

give DCR soln

variables

$$\sum_{\substack{(K,D):\, D \in \mathcal{D}_W(K) \\ j \in D}} \bar{x}_K\, p_D^K \geq 1 \quad \forall j \in W \quad c_j$$

I

$$\sum_{D \in \mathcal{D}_W(K)} p_D^K \leq 1 \quad \forall K \quad y_K$$

$$p \geq 0$$

dual $\Downarrow$

$\max \quad \sum y_K - \sum c_j$

dual $\Downarrow$

$$\min \quad \sum_K y_K - \sum_j c_j$$

$$\text{s.t.} \quad y_K - \sum_{j \in D} x_K c_j \geq 0 \qquad \forall K, D \in \mathcal{D}_U(K)$$

II $\longrightarrow$ $y_K \geq x_K \cdot \sum_{j \in D} c_j$

(*) $\Longrightarrow$ $y_K \geq x_K \cdot \max_{D \in \mathcal{D}_U(K)} c(D) \geq \bar{x}_K \cdot \text{drop}_W(K)$

Farkas $\Rightarrow$ I inf. then $\exists\, c, y$ feas. for II s.t.

$$\sum_K y_K < \sum_j c_j$$

$\overset{(*)}{\Longrightarrow}$ $c(W) > \sum_K \bar{x}_K \cdot \text{drop}_W(K)$ $\quad$ ⫽ to Bridge Lemma ▨

Note: W stays unchanged in algo but $D \subset R$ is constantly recomputed.
$\Rightarrow$ Have to recompute p^K always to ensure that U-edges are marked with prob $\geq \frac{1}{U}$

Let $\tilde{U} \subseteq U$ let $X(\tilde{U})$ be 1st iteration when all edges in $\tilde{U}$ are marked.

Lemma $\quad E[X(\tilde{U})] \leq H_{|\tilde{U}|} \cdot M$

Pf: Let m_q be best possible upper bound on $E(X(\tilde{U}))$ when $q = |\tilde{U}|$.

$q=1$ $\quad$ in each iteration, the only edge in $\tilde{U}$ is marked with prob $\geq 1/M$

$\Rightarrow m_1 \le M$

$\underline{q>1}$ λ_i : prob. that at least i edges in $\tilde{U}$ are marked in an iteration

$\Rightarrow \lambda_0 = 1 \quad \lambda_{q+1} = 0$

and $\displaystyle\sum_{i=0}^{q} \lambda_i \overset{\text{lin of exp.}}{\geq} q/M$

Condition on # i of marked edges in first iteration.

$$m_q \le 1 + \sum_{i=0}^{q} \underbrace{\Pr[\text{mark exactly } i]}_{\lambda_i - \lambda_{i+1}} \cdot \overset{\text{Exp \# rounds to make remain.}}{m_{q-i}}$$

$$\overset{\text{ih}}{\le} 1 + M \sum_{i=1}^{q} (\lambda_i - \lambda_{i+1}) \cdot H_{q-i}$$

$$+ (1-\lambda_1) m_q \quad \leftarrow \ i=0 \text{ term}$$

$$= 1 + M \sum_{i=1}^{q} \lambda_i \underbrace{(H_{q-i} - H_{q-i+1})}_{\le -1/q}$$

$$+ \lambda_1 H_q M + (1-\lambda_1) m_q$$

$$\le 1 - \frac{1}{q} M \underbrace{\sum_{i=1}^{q} \lambda_i}_{\geq q/M} + \lambda_1 H_q M + (1-\lambda_1) m_q$$

$$\le \lambda_1 H_q M + (1-\lambda_1) m_q$$

$$\Rightarrow \quad m_q \le \lambda_1 H_q M + (1-\lambda) m_q \overset{\lambda_1 > 0}{\Rightarrow} m_q \le H_q M$$

$\forall e \in S^*$, let $t(e) = \max\{t : e \in S^t\}$.

$$E[t(e)] = \sum_{q=1}^{k_e} \underbrace{\Pr(|U(e)| = q)}_{} \cdot \underbrace{E(t(e) \mid |U(e)| = q)}_{}$$

$k_e \leftarrow$ dist from root in S^*

$$E[t(e)] = \sum_{q=1} \underbrace{\Pr(|U(e)|=q)}_{\le \frac{1}{2^q}} \cdot \underbrace{E(t(e) \mid |U(e)|=q)}_{\le H_q \cdot M}$$

$$\le M \cdot \sum_{q\ge 1} \frac{1}{2^q} H_q$$

Reorder sums
$\frac{1}{q}$ appears for $\frac{1}{2^i}$, $i \ge q$

$$= M \sum_{q\ge1} \frac{1}{q} \sum_{i\ge 0} \left(\frac{1}{2}\right)^{q+i} = M \sum_{q\ge1} \frac{1}{q} \frac{1}{2^q} \cdot \underbrace{\sum_{i\ge 0}\left(\frac{1}{2}\right)^i}_{=2}$$

$$= M \sum_{q\ge 1} \frac{1}{q} \left(\frac{1}{2}\right)^{q-1} = \ln(4) M$$

Expected cost of solution

$$E\Big[\sum_{t\ge1} c(K^t)\Big] \le \frac{1}{M}\sum_{t\ge1} E[opt^t_{DCR}]$$

$$\le \frac{1}{M} \sum_{t\ge1} E[c(S^t)]$$

$$= \frac{1}{M} \sum_{e\in S^*} E[t(e)] \cdot c_e$$

$$\le (\ln 4)\cdot opt \qquad \blacksquare$$