Steiner Trees: Hypergraphic LPs

Tuesday, April 19, 2016 11:14 AM

Undirected cut LP as seen earlier

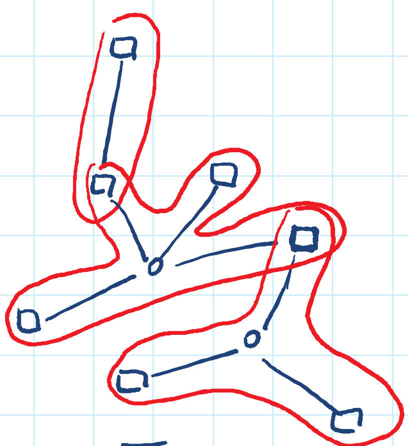
(P) $\min \; c^T x$

$$\text{s.t.} \quad x(\delta(S)) \geq 1 \quad \forall\, S \subseteq V,\ S \cap R \neq \emptyset,\ R \setminus S \neq \emptyset$$
$$x \geq 0$$

<u>Have seen:</u> $\displaystyle \max_{\text{StT instance } I} \frac{opt_I}{opt_I^P} \to 2 \text{ as } n \to \infty$

opt val. of LP (P)

Let us find a stronger LP.



T

<u>Full Component</u> maximal connected subgraph of a Steiner tree

(a) whose leaves are terminals and

(b) whose internal nodes are Steiner

<u>Immediate</u> Every Steiner tree can be split into full comps by "splitting" internal terminals

full comp

For a fc K, let $c(K)$ be cost of its edges. Natural <u>idea:</u> Construct hypergraph H with

(i) vertices R, and

(ii) one edge for every $K \subseteq R$

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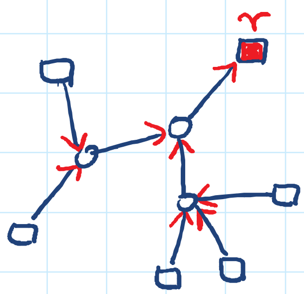
let $c(K)$ = cost of min fc spanning K
(∞ if none exists)

Steiner tree $\xRightarrow[\text{to}]{\text{reduces}}$ find mincost spanning sub-hypergraph in H

Note: all known algos for Steiner trees with apx < 2 use hypergraphic reduction in one way or another.

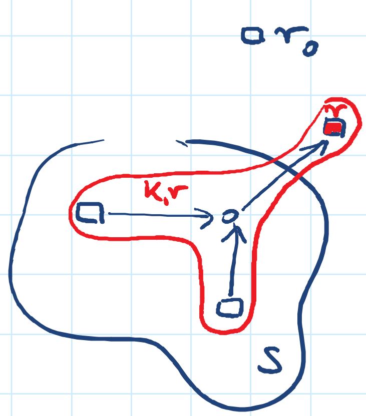
Natural LP



We will orient fc's K. Pick $r \in K$, the oriented fc (K,r) should be imagined as a directed version of K where all edges are oriented towards r.
Call r the sink of (K,r). Let $\mathcal{K}$ be the set of all directed full components.

LP has one variable for each $(K,r) \in \mathcal{K}$.
Pick root $r_0 \in R$ arbitrarily and let

$$\emptyset \neq S \subseteq R - r_0$$



Let T be any Steiner tree and orient its edges towards r_0. There must be a full component (K,r) in T that crosses S:
$K \cap S \neq \emptyset$, $r \notin S$.

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$$\Delta^+(S) = \{ (K,r) : K \cap S \neq \emptyset, r \notin S \}$$

(DCR) $\quad \min \sum_{(K,r) \in \mathcal{K}} x_{K,r} \cdot c(K)$

s.t.

$$\sum_{(K,r) \in \Delta^+(S)} x_{K,r} \geq 1 \quad \forall S \subseteq R - r_0$$

$$x \geq 0$$

Note: (DCR) has exp. many variables and constraints, and cannot (easily) be solved via Ellipsoid.
in fact...

Thm 6 (Goemans, Olver, Rothvoss, Zenklusen '12) It is NP-hard to solve (DCR)

Let $\mathcal{K}_q \subseteq \mathcal{K}$ be the collection of oriented full comps with at most $q \in \mathbb{N}$ terminals.
Let (DCR_q) be the LP obtained from (DCR) by dropping variables of fc not in $\mathcal{K}_q$.

Clearly $\quad dcr_q^{LP} \geq dcr^{LP}$ $\quad$ opt val of LP corresp. to DCR

Thm 7 [Borchers, Du '97] $\quad dcr_q^{LP} \leq \left(1 + \frac{1}{\lfloor \log_2 q \rfloor}\right) dcr^{LP}$

Won't prove this here. **Idea:** choose q large enough
[illegible]

Won't prove this here. Idea: choose q large enough but fixed s.t. $(1+\frac{1}{\lfloor \log_2 q \rfloor}) \leq 1+\epsilon$ for any given $\epsilon > 0$.

Can solve DCR_q in poly-time ($\rightarrow$ Byrka et al. '11)

From now on ignore this issue and assume that we can solve (DCR)

Goal: Solve (an approx of) (DCR) and randomly round it into a good integral soln.

Preliminary insights

Note: may assume that undirected graph G underlying StT problem is complete
(if $uv \notin E(G)$ then add it and let its cost c_{uv} equal min-cost u,v-path)

similar: may assume that c is metric:
$$c_{uv} \leq c_{uw} + c_{wv} \quad \forall u,v,w$$

Let $\bar{x}$ be a solution for (DCR).

terminal spanning tree

Thm 8 Let T be a mincost spanning tree in $G[R]$. Then
$$c(T) \leq 2 \cdot c^T \bar{x}$$

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$\Longrightarrow$ T is a 2-apx Steiner tree

Pf: Idea: Let $D = (R, A)$ be obtained from $G[R]$ by replacing each edge uv by (u,v) and (v,u) of same cost.

bidirected cut relax.

(P2) $\min \sum_a c_a x_a$

s.t.

$$x(\delta^+(S)) \geq 1 \quad \forall S \subseteq R - r_0$$

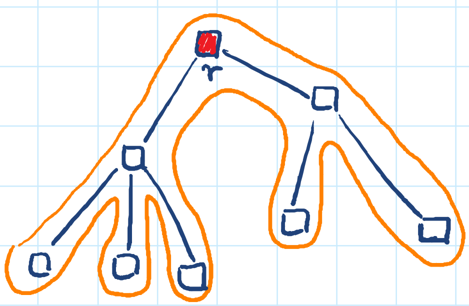
$$x \geq 0$$

Thm 3 $\Rightarrow$ (P2) is integral and $opt_2 = c(T)$

From $\bar{x}$ we construct feasible soln y for (P2) s.t. $c^T y \leq 2 c^T \bar{x}$

$$\Longrightarrow c(T) = opt_2 \leq c^T y \leq 2 c^T x$$

Let $(k, r) \in supp(x)$ (i.e., $x_{k,r} > 0$).



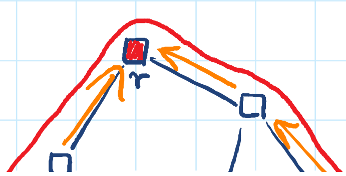
Consider tour "around" k.

$\rightarrow \tau$ Clear: $c(\tau) \leq 2c(k)$

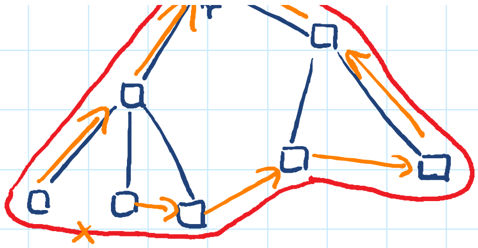
shortcut $\tau \Rightarrow \tau'$

because of Δ-inequality we have

$$c(\tau') \leq c(\tau)$$



Delete one edge of τ' and let T be resulting



Delete one edge of τ' and let T be resulting terminal spt.

Orient edges towards $r \to \vec{T}$ ▨

Let $y_a += x_{K,r}$ $\forall a \in \vec{T}$. Repeat $\forall (K,r) \in \mathrm{Supp}(x)$

<u>Claim</u> y feasible for (P_2)

<u>Pf idea</u> Consider $S \subseteq R - r_0$.

<u>Know</u>:

$$x(\Delta^+(S)) \geq 1 \Longrightarrow y(\delta^+(S)) \geq 1$$ ▨

▨

Natural idea for rounding algo

Repeat

(1) Solve (DCR) $\longrightarrow \bar{x}$, let $M = \sum_{(K,r)} \bar{x}_{K,r}$

(2) Sample (K,r) with prob $\frac{\bar{x}_{K,r}}{M}$ and add K to solution

(3) Identify terminals in K

In the analysis we need to quantify the "drop" in mst(G[R]) in every contraction step (3).

Let S be a Steiner tree spanning R, and let K be some full component.

and let K be some full component.
Collapse the terminals of K into a single node in $G[S]$ and let resulting multigraph be S/K

<u>note:</u> MST S' of S/K has $|S|-(|K|-1)$ edges.

$$\longrightarrow \quad \mathrm{Drop}_S(K) = S \setminus \mathrm{MST}(S/K)$$
$$= \operatorname{argmax}\{c(D) : D \subseteq S,\ S \setminus D \cup \tbinom{K}{2} \text{ connects } V(S)\}$$

$$\mathrm{drop}_S(K) = c(\mathrm{Drop}_S(K)) = c(S) - \mathrm{mst}(S/K)$$

Bridge Lemma [BGRS'11] $\quad c(T) \le \sum_{(K,r)} \mathrm{drop}_T(K)\, x_{K,r}$

for any terminal spt T, and feasible DCR soln x.

<u>First:</u> how to use the

Let $\bar{x}$ be soln to DCR and T a min terminal spt.
We sample fc K according to $\bar{x}$, contract K and recompute terminal spt $\to T'$.

$$E(c(T')) = c(T) - E[\mathrm{drop}_T(K)]$$

$$= c(T) - \frac{1}{M} \sum_{K,r} \bar{x}_{K,r}\, \mathrm{drop}_T(K)$$

$$= c(T) - \frac{1}{M} \underbrace{\sum_{K,r} \bar{x}_{K,r}\, \mathrm{drop}_T(K)}_{\geq c(T)}$$

$$\leq \left(1-\frac{1}{M}\right) c(T)$$

$$\underset{\text{Thm 8}}{\leq} \left(1-\frac{1}{M}\right) 2 \cdot \mathrm{opt}_{DCR}$$

Note: Contracting terminal vertices in K creates terminal set $R' = R/K$ and

$$\mathrm{opt}_{DCR(R')} \leq \mathrm{opt}_{DCR}$$

So, repeating the contraction algorithm ℓ times, the expected cost of the terminal spanning tree T' of the resulting instance is

$$E(c(T')) \leq \left(1-\frac{1}{M}\right)^{\ell} \cdot 2 \cdot \mathrm{opt}_{DCR}$$

1st round LP cost

note: we assumed

$\mathbb{1}^T x = M$ for feasible sol. x to DCR

M may not be same throughout

workaround: choose $M = \#$ variables

$\Longrightarrow \mathbb{1}^T x \leq M \quad \forall$ minimal solns to DCR

In rounding: do nothing with prob $1 - \frac{\mathbb{1}^T x}{M}$.

Choose $\ell = \ell_0 \cdot M$ for some $\ell_0 \in \mathbb{N}$

Suppose we contract $K_1, \dots, K_\ell$ in the process, in order to arrive at terminal spt. T of $R/K_1/\dots/K_\ell$.

in order to arrive at terminal spt. T of $K, K_1, \dots, K_\ell$.

Have: (i) $E(c(T)) \le (1-\frac{1}{M})^{\ell_0 M} \, 2\, \mathrm{opt}_{DCR}$

$$\le 2e^{-\ell_0}\, \mathrm{opt}_{DCR}$$

(ii) $E(c(K_i)) \le \frac{1}{M} \sum_{(K,r)} x_{K,r} \cdot c(K) \le \frac{1}{M}\mathrm{opt}_{DCR}$

$$\Longrightarrow \sum_{i=1}^{\ell} E[c(K_i)] \le \ell_0 \cdot \mathrm{opt}_{DCR}$$

note: $\underbrace{T + K_1 + \dots + K_\ell}_{\bar{T}}$ is a feasible Steiner tree

$$E[c(\bar{T})] \le \underbrace{(2e^{-\ell_0} + \ell_0)}\, \mathrm{opt}_{DCR}$$

Choose $\ell_0 = \ln 2 \Longrightarrow 1 + \ln 2 \approx 1.694$

Thm 9 DCR has a gap of at most 1.694.

$\rightarrow$ can be improved to $1 + \frac{\ln 3}{2} \approx 1.55$ quite easily.
[BGRS'11, CKP'10]

and also $\ln 4 \approx 1.39$ known [GORZ'12] challenging

Todo prove bridge lemma

Let S be a Steiner tree on R. Define the following auxiliary cost function

$w : R \times R \to \mathbb{R}_+$

w_{uv} : max-cost of any edge on u,v-path in S

Lemma 1 S: Steiner tree, w: assoc. aux cost func.
For any $K \subseteq R$ there is a tree $Y \subseteq R' \times R'$ s.t.

(a) Y spans K

(b) $w(Y) = \mathrm{drop}_S(K)$

(c) For any $uv \in Y$, u,v-path in S has unique edge from $\mathrm{Drop}_S(K)$

Pf:

Recall

identify K in S

$$\mathrm{Drop}_S(K) = S \setminus \mathrm{MST}(S/K)$$

$$= \{e_1, \dots, e_{p-1}\} \qquad p = |K|$$

$S \setminus \{e_1, \dots, e_{p-1}\}$ is forest of trees

$$T_1, \dots, T_p$$

where each T_i contains exactly one terminal $r_i \in K$.

① T_i cannot contain $u, v \in K$ as $S/K \cup T_i$ has cycle

② $T_i \cap K = \emptyset \Rightarrow \mathrm{MST}(S/K)$

not connected ⌋

Suppose that e_i connects trees $T_{i'}$ and $T_{i''}$.

The add $r_{i'} r_{i''}$ to Y.

$\Longrightarrow$ Y has p nodes and $p-1$ edges

Claim: Y is acyclic

Pf:

Suppose not and $\exists$ cycle

$$v_1 v_2, \ldots, v_q v_1$$

in Y. Suppose that $v_i \in T_{i'}$ and let $e_{i'} \in \mathrm{Drop}_S(k)$

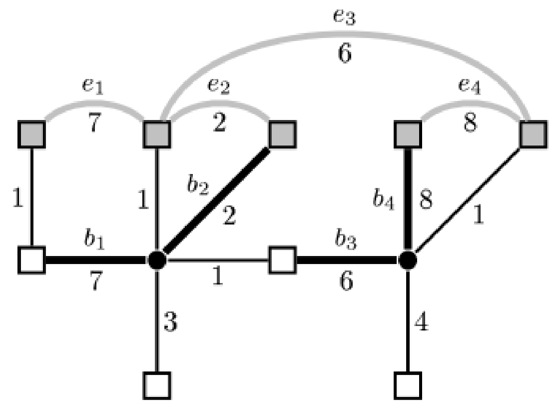
$$\Longrightarrow T_1 \cup \ldots \cup T_p \cup \{e_{1'}, \ldots, e_{q'}\} \subseteq S$$

but cyclic ↯ ■

Note: e_i is unique $\mathrm{Drop}_S(k)$ edge on $r_{i'}, r_{i''}$-path in S → (c)

$$\Longrightarrow w_{r_{i'} r_{i''}} := c_{e_i} = \max_{e \in P_i} c_e \rightarrow \text{(b)}$$

■



[Byrka et al. '11]

Proof of Bridge Lemma

Idea: construct auxiliary graph $H = (R, F)$ with new edge-costs $w : F \to \mathbb{R}_+$ and construct bidirected cut relaxation (BCR) for H:

obtain $D = (R, A)$ from H by adding $(u,v), (v,u)$ $\forall\, uv \in F$ with cost w_{uv}. Pick $r \in R$.

BCR $\quad \min \sum_{a \in A} w_a x_a$

$$\text{s.t.} \quad x(\delta^+(U)) \geq 1 \quad \forall\, U \subseteq R \setminus r$$
$$x \geq 0$$

① Pick $y \in \mathbb{R}_+^A$ feasible for BCR s.t.

$$w^T y = \sum_{K,r} \mathrm{drop}_T(K)\, x_{K,r}$$

—

② Every spt. of H has weight $\geq c(T)$

Edmond's thm : DCR is integral

$$\Longrightarrow c(T) \leq U^T y = \sum_{K,r} \mathrm{drop}_T(K)\, x_{K,r}$$

① $y = 0$ initially

<u>For each</u> $(K,r) \in \mathrm{Supp}(x)$

- Use Lemma ⊛ to construct spt. Y_K for K and weight w
- add Y_K to H
- Orient Y_K towards r $\longrightarrow Y_{K,r}$
  add $x_{K,r}$ to y_a $\forall a \in Y_{K,r}$

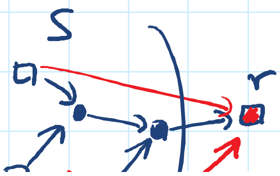
$$\longrightarrow \sum_{a \in Y_{K,r}} w_a y_a = x_{K,r} \underbrace{\sum_{a \in Y_{K,r}} w_a}_{\text{Lemma 1 : } \mathrm{drop}_T(K)}$$

$$\Longrightarrow \sum_{a \in H} w_a y_a = \sum_{(K,r)} x_{K,r}\, \mathrm{drop}_T(K) \qquad (1)$$

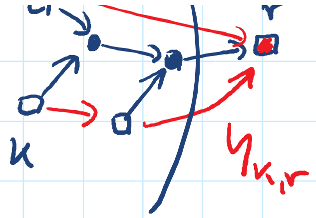
<u>To show</u> y is feasible for (BCR)

Let $S \subseteq R \setminus r$. Since x feas. for DCR

$$\Longrightarrow x(\Delta^+(S)) \geq 1$$



Each $(K,r) \in \Delta^+(S)$



Each $(k,r) \in \Delta^+(S)$ contributes at least $x_{k,r}$ to $y(\delta^+(S))$ ✓

Finally: any spt in H has cost $\geq c(T)$.

Add the edges of T to H, and let $w_e = c_e$ $\forall e \in T$.

Claim: T is a min w-weight spt in H

Pf: ETS: $\forall uv \in E(H) \setminus T$

$$w_{uv} \geq \max_{e \in P_{uv}} w_e \qquad \circledast$$

where P_{uv} is unique u,v-path in T.

But note: $uv \in E(H) \setminus T$ then we set

$$w_{uv} = \max_{\substack{e \in u,v\text{-path } P \\ \text{in } T}} c_{uv}$$

$\Longrightarrow \circledast$ holds ▨