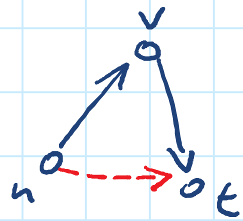
Directed Splitting Off [Frank'89]

Tuesday, April 12, 2016 11:46 AM

First: (directed) splitting off Useful technique to reduce #arcs in given digraph $D = (N, A)$ while maintaining local connectivity.

$v \in N$, $(u,v), (v,t) \in A$



Splitting off $(u,v), (v,t)$:

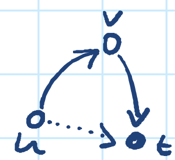
replace $(u,v), (v,t)$ by (u,t)

Theorem 5 [Frank '89]

$D = (N, A)$

Eulerian, $v \in N$ ← $d^-(v) = d^+(v)$ $\forall v \in N$

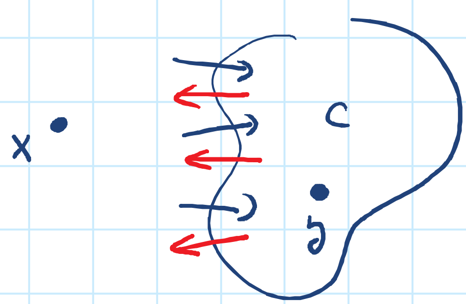
$(v,t) \in A$



$\exists (u,v)$ s.t. (u,v) & (v,t) can be split off w/o affecting $\lambda_D(x,y)$ $\forall x,y \in N - v$

~~Pf~~

Call $C \subseteq N$ critical in D for $x \in N \setminus C$, $y \in C$ and $\lambda_D(x,y) = d^-(C)$



Note: $d^+(C) = d^-(C)$ as D Eulerian

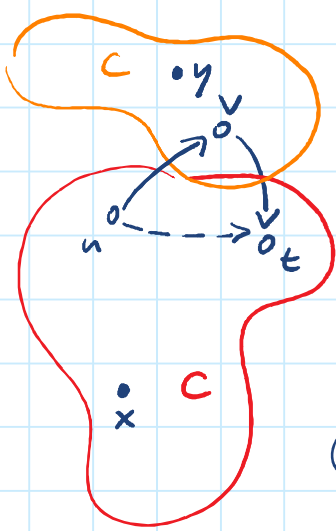
$$\left[0 = \sum_{v\in C}\left(d^+(v) - d^-(v)\right) = d^+(C) - d^-(C)\right]$$

Let $G = (N,E)$ be underlying undirected graph of D.
Then $C \subseteq N$ is critical for G and $x,y \in N$ if

$$\lambda_G(x,y) = d(C)$$
and
$$|\{x,y\} \cap C| = 1$$

Note: $\lambda_G(x,y) = 2\lambda_D(x,y)$

and C is critical in G for x,y iff
C is critical in D for x,y



splitting off (u,v) & (v,t) not possible if
$\exists\, C \subseteq V$ that is critical and loses an arc by splitting off (u,v) & (v,t).
So:

① $\exists$ critical set C s.t. $(v,t) \in \delta^-(C)$ and $(u,t) \notin \delta^-(C)$, or

② $\exists$ critical set C s.t. $(u,v) \in \delta^-(C)$ and $(u,t) \notin \delta^-(C)$
$\Rightarrow$ C critical for x,y
$\Rightarrow$ $V \setminus C$ critical for y,x

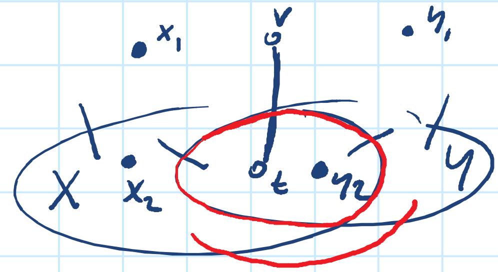
$\Longrightarrow$ ① suffices

<u>So:</u> if $\nexists$ critical set C s.t. $(v,t) \in \delta^-(C)$ then <u>any</u> (u,v), (v,t) can be split-off.

<u>Claim</u> $\exists$ unique maximal critical set C s.t. $(v,t) \in \delta^-(C)$

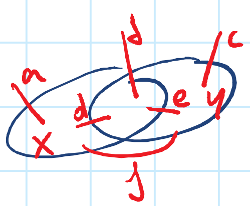
<u>Pf:</u> Sp. not and there exist 2 such sets X, Y. Then X, Y are critical also in G.

Suppose X critical f. $x_1, \overset{\in X}{x_2}$ and Y for $y_1, \underset{\in Y}{y_2}$.



First argue that
$$\{x_2, y_2\} \cap X \cap Y \neq \emptyset.$$

Suppose not then $x_2 \in X \setminus Y$, $y_2 \in Y \setminus X$, and



$$\underbrace{d(X)}_{a+b+e+f} + \underbrace{d(Y)}_{b+c+d+f} = \underbrace{d(X \setminus Y)}_{a+d+f} + \underbrace{d(Y \setminus X)}_{c+e+f} + 2d(X \cap Y, \overline{X \cup Y})$$

$$\geq \lambda_G(x_1, x_2) + \lambda_G(y_1, y_2)$$

$$= d(X) + d(Y)$$

$$\Longrightarrow 2d(X \cap Y, \overline{X \cup Y}) = 0$$

not true here!

not true here!

$$d(X) + d(Y) = \underbrace{d(X \cap Y)}_{b+d+e} + \underbrace{d(X \cup Y)}_{a+b+c} + \underbrace{2\, d(X \setminus Y, Y \setminus X)}_{2f}$$

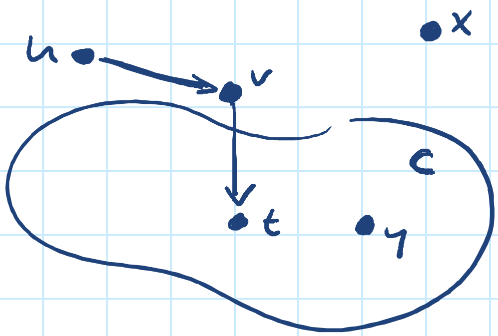
$$\geq d(X) + d(Y)$$

$\Longrightarrow$ $X \cup Y$ is also critical. $\blacksquare$

So now let C be an inclusion-wise maximal critical set s.t. $(v,t) \in \delta^-(C)$. Supp C is critical for (x,y)

$x \notin C$, $y \in C$

$x \neq v$.



Note that there must be $(u,v) \in A$ s.t. $u \notin C$. Otherw.

$$\lambda_D(x,y) = d^-(C) > d^-(C+v)$$
$$\geq \lambda_D(x,y) \quad ↯$$

$\Longrightarrow$ can split-off $(u,v), (v,t)$ $\blacksquare$

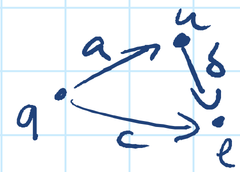
Now prove Dang-Jensen et. al (Thm 2).

Pf of Thm 2. Assume $|N| \geq 2$ (other trivial)

Proof by induction on # arcs in D.

Pick a node $u \in N \setminus r$ with smallest λ_u.

Thm 3
$\Longrightarrow \exists\, a \in \delta(\bar{u}), b \in \delta(u)$ that can be split off



Let $D' = D \setminus a, b \cup c$.

In D' find arb. $A_1, \ldots, A_p$ s.t. $v \neq u$ is in λ_v many

and u is in $\lambda_{D'}(r,u) \geq \lambda_D(r,u) - 1$ many.

Note: Suppose A_1 contains c (wlog, why?)

Case 1: A_1 does not contain u

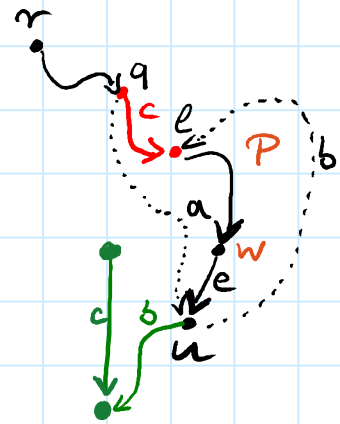
$\Longrightarrow$ replace c by a, b

The resulting arb. spans u as well, and u is now in λ_u arbs.

Case 2: A_1 does contain u

Case 2: A_1 does contain u

Consider r,u-path P in A_1.



Case 2.1 $c \notin P$

Delete c from A_1 and add arc b.

spare arc: a

Case 2.2: $c \in P$

Let e be the arc on P with head u.
Delete c and e from A_1 and add a and b.

spare arc: e

We are now done if u is in λ_u arbs.
If not then u is in $\lambda_u - 1$ arbs and
Case 2 must have applied.

Note also: by choice of u, all nodes
$v \in N \setminus u$ must be contained in $\lambda_v \geq \lambda_u$
arborescences from $A_1, \ldots, A_p$

Case 2.1 let A_i be an arb that

contains q but not u

$\Longrightarrow$ add spare arc a to A_i;

<u>Case 2.2</u> Suppose $e = (w,u)$ and let A_i be an arc that contains w but not u

$\Longrightarrow$ add spare arc e to A_i;

$\blacksquare$