Lift & Project

Monday, July 11, 2016 4:00 PM

<u>Have seen:</u> strengthening LP relaxations by adding strong valid inequalities.

<u>E.g.:</u> Knapsack (→ Carnes & Shmoys)
adding flow cover inequalities reduces gap to ≤ 2

<u>in fact:</u> the gap of the strengthened LP is ~ 2 (see Carr et al. '00)

There is of course a PTAS for max- and min Knapsack. Having an LP with gap 2 not impressive.

<u>Today:</u> Lift-and Project hierarchies
- strengthening weak LPs systematically
- <u>Idea:</u> (i) find valid relaxation of given problem in higher-dim space "lift"

  (ii) projection to native space is tighter relaxation of problem "project"

There are many Lift & Project hierarchies, we focus on Lasserre that produces SDP relaxations.

<u>Positive Def. Matrices</u>

Symmetric matrix M is positive semidef (psd) if $x^T M x \geq 0 \;\; \forall x$

<u>Equiv:</u> (i) any principal submatrix has

$\det \geq 0$

(ii) $\exists$ vect. v_i s.t. $M_{ij} = \langle v_i, v_j \rangle$

Round-t Lasserre relax

linear relax of binary problem

<u>Given</u> $K = \{ x \in \mathbb{R}^n : Ax \geq b \}$ A $m \times n$ mat.

Introduce new variables y_I $\forall I \subseteq [n]$, $|I| \leq 2t+1$

<u>Intuition</u> think of y_I as representing event

$$\bigwedge_{i \in I} (x_i = 1)$$

want: $y_\emptyset = 1$ and $y_{\{i\}} = x_i$

$\boxed{\mathrm{Las}_t(K)}$

$(M_t(y))$

$$\boxed{(y_{I \cup J})_{|I|,|J| \leq t}} \succeq 0$$

$$\boxed{\Big(\sum_{i \in [n]} A_{\ell,i} \cdot y_{I \cup J \cup \{i\}} - b_\ell y_{I \cup J} \Big)_{|I|,|J| \leq t}} \succeq 0 \quad \forall \ell \in [m]$$

$(M_t^\ell(y))$

$$y_\emptyset = 1$$

Consistency — $\boxed{M_t(y)}$ moment matrix

ensures constraints — $\boxed{M_t^\ell(y)}$ moment matrix of slacks

$\mathrm{Las}_t^{proj}(K) = \{ (y_{\{1\}}, \ldots, y_{\{n\}}) : y \in \mathrm{Las}_t(K) \}$

projection of $\mathrm{Las}_t(K)$ onto original space.

<u>Note</u>: set of PSD matrices of certain dim form polyhedral cone

$\Longrightarrow$ $Las_t(K)$ is polyhedron

Separation $\Longrightarrow$ computing neg. EV of involved mat if that exists

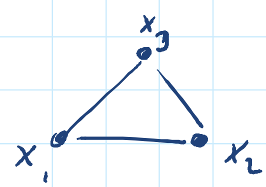
Can be done in poly time

$\Longrightarrow$ can (approx.) optimize over $Las_t(K)$ in $n^{O(t)} m^{O(1)}$ time.

Ex: Independent set $G = (V, E)$

$$K = \{ x \in \mathbb{R}^n : x_u + x_v \le 1 \ \forall uv \in E$$
$$\mathbb{0} \le x \le \mathbb{1} \}$$

$$M_1(y) = \begin{pmatrix} 1 & y_1 & y_2 & y_3 \\ y_1 & y_1 & y_{21} & y_{31} \\ y_2 & y_{21} & y_2 & y_{32} \\ y_3 & y_{31} & y_{32} & y_3 \end{pmatrix}$$



$$M_1^{12} = \begin{pmatrix} 1 - y_1 - y_2 & y_1 - y_1 - y_{12} & \cdots \\ \vdots & y_1 - y_1 - y_{12} & \\ \vdots & & \ddots \end{pmatrix}$$

Basic Properties

(P1) $\mathrm{conv}(K \cap \{0,1\}^n) \subseteq Las_t^{proj}(K) \quad \forall t \ge 1$

given $x \in K \cap \{0,1\}^n$, let

$$y_I = \begin{cases} 1 & : x_i = 1 \ \forall i \in I \\ 0 & : \text{oth.} \end{cases}$$

note: $(M_t(y))_{I,J} = y_{I \cup J} = y_I \cdot y_J = (yy^T)_{IJ}$

and $yy^T \succeq 0$

Similarly $M_t^\ell(y) \succeq 0 \quad \forall \ell$

So: y feas. for $Las_t(K)$ ✓

(P2) $y \in Las_t(K)$ Then

(a) $\underbrace{(y_1, \dots, y_n)}_{\bar{y}} \in K$

$$M_t^\ell(y)_{\emptyset,\emptyset} : \sum_{i=1}^{n} A_\ell y_i - b_\ell \underbrace{y_\emptyset}_{=1} \overset{\text{psd}}{\geq} 0$$

so: $\bar{y}$ sat. all constraints

also: consider principal subm. def. by $\emptyset$ and I of $M_t(y)$:

$$\begin{bmatrix} y_\emptyset & y_I \\ y_I & y_I \end{bmatrix} = A$$

$$\det(A) = y_I(1 - y_I) \overset{\text{psd}}{\geq} 0$$

$$\Longrightarrow \quad 0 \leq y_I \leq 1 \quad \forall I$$

(b) $y_J \leq y_I$ for $I \subseteq J$, $|I| \leq |J| \leq t$

Consider submatrix for I, J

$$\left| \begin{bmatrix} y_I & y_J \\ y_J & y_J \end{bmatrix} \right| = y_I y_J - y_J^2$$

$$= y_J(y_I - y_J) \geq 0$$

$$\Longrightarrow \quad y_J \leq y_I$$

Lemma 1 $y \in Las_t(K)$ $i \in [n]$, $0 < y_i < 1$

$\Longrightarrow y \in conv\{ z \in Las_{t-1}(K) \mid z_i \in \{0,1\}\}$

Pf: Define

$$z_I^{(1)} = \frac{y_{I \cup \{i\}}}{y_i} \qquad z_I^{(0)} = \frac{y_I - y_{I\cup\{i\}}}{1-y_i}$$

$$\Longrightarrow \quad y = y_i z_I^{(1)} + (1-y_i) z_I^{(0)}$$

and $$z_i^{(1)} = \frac{y_i}{y_i} = 1, \quad z_i^{(0)} = \frac{y_i - y_i}{1-y_i} = 0$$

ToDo show psd constraints

$$y \in Las_t(K) \Longrightarrow M_t(y) \succeq 0$$

$$\Longrightarrow \exists\, v_I \text{ s.t. } M_t(y)_{IJ} = \langle v_I, v_J \rangle$$

$\Longrightarrow$ for $I, J \subseteq [n]\setminus i$, $|I|, |J| < t$: $$\left\langle \frac{v_{I\cup i}}{\sqrt{y_i}}, \frac{v_{J\cup i}}{\sqrt{y_i}} \right\rangle = \frac{1}{y_i} \cdot y_{I\cup J\cup i} = z^{(1)}_{I\cup J}$$

$$\Longrightarrow \quad M_{t-1}(z^{(1)}) \succeq 0$$

Also: $$\left\langle \frac{v_I - v_{I\cup i}}{\sqrt{1-y_i}}, \frac{v_J - v_{J\cup i}}{\sqrt{1-y_i}} \right\rangle$$

$$= \frac{v_I v_J - v_I v_{J\cup i} \overbrace{- v_J v_{I\cup i} + v_{I\cup i} v_{J \cup i}}^{0}}{1-y_i}$$

$$= \frac{y_{I\cup J} - y_{I\cup J \cup i}}{1-y_i} = z^{(0)}_{I\cup J}$$

$$\Longrightarrow \quad M_{t-1}(z^{(0)}) \succeq 0$$

Similar for slack matrices. ■

Immediate consequence via induction:

Corollary 1 $y \in Las_t(K)$, $S \subseteq [n]$, $|S| \leq t$

$$\Longrightarrow y \in conv\{z \in Las_{t-|S|}(K) \mid z_i \in \{0,1\} \; \forall i \in S\}$$

Actually can get an explicit convex combination.
Let

(Def) $$y_I^{J_0 J_1} = \sum_{H \subseteq J_0} (-1)^{|H|} y_{I \cup J_1 \cup H}$$

Lemma 2 $y \in Las_t(K)$, $S \subseteq [n]$

$$\Longrightarrow y = \sum_{\substack{J_0 \cup J_1 = S \\ y_\emptyset^{J_0 J_1} > 0}} y_\emptyset^{J_0 J_1} \underbrace{\left(\frac{y^{J_0 J_1}}{y_\emptyset^{J_0 J_1}} \right)}_{z^{J_0 J_1}} \quad (*)$$

EX
check that
$\sum y_\emptyset^{J_0 J_1} = 1$

and $z^{J_0 J_1} \in Las_{t-|S|}(K)$, and

$$z_i^{J_0 J_1} = \begin{cases} 1 & : i \in J_1 \\ 0 & : i \in J_0 \end{cases}$$

Proof: fairly straightforward via induction and Lemma 1.

(→ Scribbles)

Note: Suppose $y \in Las_n(K)$ then there is a random variable $X \in K \cap \{0,1\}^n$ with

$$(**) \qquad Pr\left[\bigwedge_{i \in I} X_i = 1\right] = y_I$$

Simply take decomposition of Lemma 2
and draw $z^{\partial_0 \partial_1}$ with prob

$$y^{\partial_0 \partial_1}$$

$\Longrightarrow$ (**) holds.

Inclusion-Exclusion:

$$\Pr\Big[\bigvee_{i \in \partial_0} (X_i = 1)\Big] = \sum_{\emptyset \subset H \subseteq \partial_0} (-1)^{|H|+1} \Pr\Big(\bigwedge_{i \in H} (X_i = 1)\Big)$$

$$\Longrightarrow \Pr\Big[\bigwedge_{i \in \partial_0} (X_i = 0)\Big] = 1 - \Pr\Big[\bigvee_{i \in \partial_0} (X_i = 1)\Big]$$

$$= \sum_{H \subseteq \partial_0} (-1)^{|H|} \Pr\Big[\bigwedge_{i \in H} (X_i = 1)\Big]$$

Intersecting all events with $\bigwedge_{i \in \partial_1} (X_i = 1)$ gives

generalized inclusion exclusion formula:

$$\Pr\Big[\bigwedge_{i \in \partial_1} (X_i = 1) \wedge \bigwedge_{i \in \partial_0} (X_i = 0)\Big]$$

$$= \sum_{H \subseteq \partial_0} (-1)^{|H|} \Pr\Big[\bigwedge_{i \in \partial_1 \cup H} X_i = 1\Big]$$

$$\overset{(**)}{=} \sum_{H \subseteq \partial_0} (-1)^{|H|} y_{\partial_1 \cup H}$$

$$\overset{(DEF)}{=} y_{\emptyset}^{\partial_0 \partial_1}$$

<u>Hence:</u>

$\partial_0 \partial_1$

$$z_I^{\partial_0,\partial_1} = \frac{y_I^{\partial\partial_1}}{y_\emptyset^{\partial_0\partial_1}}$$

$$= \frac{\Pr[\bigwedge_{i\in\partial_1\cup I}(X_i=1)\wedge\bigwedge_{i\in\partial_0}(X_i=0)]}{\Pr[\bigwedge_{i\in\partial_0}(X_i=0)\wedge\bigwedge_{i\in\partial_1}(X_i=1)]}$$

$$= \Pr\left[\bigwedge_{i\in I}(X_i=1)\,\middle|\,\bigwedge_{i\in\partial_0}(X_i=0)\wedge\bigwedge_{i\in\partial_1}(X_i=1)\right]$$

Thus: $z^{\partial_0\partial_1} \equiv y$ conditioned on ∂^0, ∂^1

This global probabilistic view holds even locally:

Lemma 3 $y \in Las_t(K)$ $S \subseteq [n]$, $|S| \leq t$

$\Rightarrow \exists$ dist $D(S)$ over $\{0,1\}^{|S|}$ w/ $\Pr_{z\sim D(S)}[\bigwedge_{i\in I}(z_i=1)] = y_I \quad \forall I \subseteq S$

Pf: again use convex comb. in Lemma 2.

note: nice consistency property:

$S \neq S' \subseteq [n]$ $|S|, |S'| \leq t$ then

$$\Pr_{z\sim D(S')}\left(\bigwedge_{i\in I}(z_i=1)\right) = \Pr_{z\sim D(S)}\left(\bigwedge_{i\in I}(z_i=1)\right) = y_I \quad \forall I \subseteq S\cap S'$$

skip?

Ex: 3-coloring

Let $G=(V,E)$ be a 3-colorable graph; i.e.,
$\exists$ coloring $(x_v)_{v\in V}$, $x_v \in \{g,r,b\}$

s.t. $\forall uv \in E: x_u \neq x_v$.

$$K = \{ x \in [0,1]^{3n} \mid x_{i,c} + x_{j,c} \leq 1 \quad \forall ij \in E, \; c \in \{g, r, b\} \}$$

<u>Lemma 4:</u> $y \in Las_{3t}(K)$

$\Rightarrow \exists$ family of dist. $(\mathcal{D}(S))_{S \subseteq V, |S| \leq t}$ s.t.

(a) each event $x \sim \mathcal{D}(S)$ is valid 3-coloring of $G[S]$

(b) $\Pr_{x \sim \mathcal{D}(S)} [x(i_1) = c_1, \ldots, x(i_k) = c_k] = y_{\{(i_1, c_1), \ldots, (i_k, c_k)\}}$

$\forall i_1, \ldots, i_k \in S, \; c_1, \ldots, c_k \in \{g, r, b\}$ $\lrcorner$

Decomposition Property

Let $K \subseteq \mathbb{R}^n$ be relax of 0,1-polyhedron. We know that $Las_n(K) = conv(K \cap \{0,1\}^n)$. But Lasserre is often stronger.

<u>Ex:</u> Knapsack $\quad K = \{ x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i \leq 1.9 \}$

v_i : value of object i

Consider $y \in Las_2(K)$ and let $0 < y_i < 1$.

<u>Claim</u> $y_{\{i,j\}} = 0 \quad \forall i \neq j$

<u>Pf:</u> Suppose not and $\exists \; i \neq j$ s.t.

$y_{\{i,j\}} > 0$

Then pick convex comb. & define:

$$y = y_i\, y^{(1)} + (1 - y_i)\, y^{(0)}$$

Know (P2b): $y^{(0)}_{\{ij\}} \leq y^{(0)}_i = 0$

$$\Longrightarrow y^{(1)}_{\{ij\}} > 0 \quad \Longrightarrow y^{(1)}_j > 0$$

But this cannot be, since by Lemma 1:
$\exists\, y^{(10)}, y^{(11)} \in \mathrm{Las}_0(K) = K$ s.t.

$$y^{(1)} = y^{(1)}_j\, y^{(11)} + (1 - y^{(1)}_j)\, y^{(10)}$$

and $y^{(11)}, y^{(10)} \in K$ but $y^{(11)}$ viol. cap constraint.

$\blacksquare$

Let $\{v_I\}_{\substack{I \subseteq [n] \\ |I| \leq 2}}$ $\mathfrak{I}$-dim vect. s.t.

$$\langle v_I, v_J \rangle = y_{I \cup J}$$
$$\Longrightarrow \langle v_i, v_j \rangle = y_{\{i,j\}} = 0 \quad \forall\, i \neq j$$

$$\Longrightarrow \sum_{i=1}^n y_i = \sum_{i=1}^n \|v_i\|_2^2 = \sum_{i=1}^n \langle v_\emptyset, v_i \rangle = \sum_{i=1} \langle v_\emptyset, \frac{v_i}{\|v_i\|_2} \rangle^2$$

$\langle v_i, v_i \rangle = \langle v_i, v_\emptyset \rangle$

$$\leq (\cos\varphi)^2\, \|v_\emptyset\|_2^2$$
$$\leq \|v_\emptyset\|_2^2 = 1$$

$K + \mathbb{1}^T x \leq 1$ defines $\mathrm{conv}(K \cap \{0,1\}^n)$.

$$\Longrightarrow \mathrm{Las}_2^{\mathrm{proj}}(K) = \mathrm{conv}(K \cap \{0,1\}^n)$$

General result:

Lemma 5 $K = \{x \in \mathbb{R}^n : Ax \geq b\}$

Suppose that any $x \in K$ has $\leq t$ ones.
Then

$$\mathrm{Las}^{\mathrm{proj}}_{t+1}(K) = \mathrm{conv}(K \cap \{0,1\}^n)$$

Pf: Know: $M_{t+1}(y) \succeq 0$. Similar argument as before shows that

$$y_I = 0$$

$$\forall I \subseteq [n] : t < |I| \leq \underbrace{2(t+1)+1}_{\text{index set dim in } \mathrm{Las}_{t+1}(K)}$$

Define: $y_I = 0 \quad \forall I, |I| > 2t+3$.

$$M_n(y) = \begin{array}{cc} \left[\begin{array}{c|c} M_{t+1}(y) & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} |I| \leq t+1 \\ \\ |I| > t+1 \end{array} \\ \begin{array}{cc} |I| \leq t+1 & |I| > t+1 \end{array} & \end{array}$$

Clearly $M_n(y) \succeq 0$ as $M_{t+1}(y) \succeq 0$

One can also show that $M_n^\ell(y) \succeq 0 \quad \forall 1 \leq \ell \leq m$.

$$\Longrightarrow \mathrm{Las}_{t+1}(K) \subseteq \mathrm{Las}_n(K) \qquad \blacksquare$$

Decomposition thm (Karlin, Mathieu, Nguyen '11)

Suppose that all $x \in K$ have $\leq k$ ones in $S \subseteq [n]$ and $k \leq t$. Then

$$y \in \mathrm{Las}_t(K) \Longrightarrow y \in \mathrm{conv}\{z : z \in \mathrm{Las}_{t-k}(K) :$$

$z_i \in \{0,1\} \quad \forall i \in S \}$