# Capacitated Facility Location

Tuesday, June 28, 2016 8:47 AM

(metric) uncapacitated facility location (UFL)

given:
- collection of potential facility and client locations $\mathcal{F}$ and $\mathcal{D}$
- metric $c$ on $\mathcal{F} \cup \mathcal{D}$
- facility opening costs $f_i \ \forall i \in \mathcal{F}$

goal:
- decide on set $\mathcal{O} \subseteq \mathcal{F}$ of facilities to open and assign each $j \in \mathcal{D}$ to facility $i(j) \in \mathcal{O}$
- minimize $\underbrace{f(\mathcal{O})}_{\text{opening cost}} + \underbrace{\sum_{j \in \mathcal{D}} c(j, i(j))}_{\text{assignment cost}}$

What's known:

[Guha, Khuller '99] no $< 1.463$-apx unless $NP \subset DTIME(n^{O(\log\log n)})$

[Li '11] $1.488$-apx

in the non-metric setting: $\Theta(\log |\mathcal{D}|)$ [Hochbaum '82]

Natural IP $\quad x_{ij} = \begin{cases} 1 & : j \text{ assigned to } i \\ 0 & : \text{otherw.} \end{cases}$

**Natural IP**

$$x_{ij} = \begin{cases} 1 & : \ j \text{ assigned to } i \\ 0 & : \ \text{oth.} \end{cases}$$

$$y_i = \begin{cases} 1 & : \ i \text{ open} \\ 0 & : \ \text{oth.} \end{cases}$$

$(P_{UFL})$

$$\min \ \sum_{ij} c_{ij} x_{ij} + \sum_i f_i y_i$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad \forall j \in D$$

$$x_{ij} \le y_i \qquad \forall i \in F$$

$$x, y \geq 0$$

$(P_{UFL})$ known to be useful in approximation
Shi Li's 1.488-apx based on rounding
solutions to LP.

**Capacitated Facility Location**

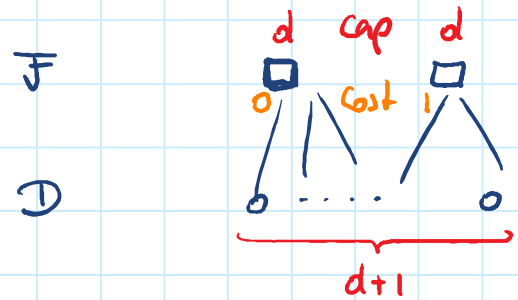
also give capacity $u_i$ $\forall i \in F$
now: assign clients to facilities
s.t. no more than $u_i$ clients
are assigned to $i$

**Candidate LP:** add $\sum_j x_{ij} \le u_i \cdot y_i \quad \forall i \in F$

$\Longrightarrow$ $P_{CFL}$

$P_{CFL}$ is not a good LP.

## Easy gap examples




$opt = 1$

$opt_{LP} = \frac{1}{d}$

unbounded integrality gap

## Strengthening this LP

1st try: Knapsack / Flow Cover inequalities

- Fix partial assignment $g : \mathcal{D}' \to \mathcal{F}$

  $(\mathcal{D}' \subseteq \mathcal{D})$

  Reasoning:

  Assuming that each $j \in \mathcal{D}'$ is assigned to $g(j) \in \mathcal{F}$, any feasible solution to CFL instance should assign clients in $\mathcal{D} \setminus \mathcal{D}'$ feasibly in residual instance

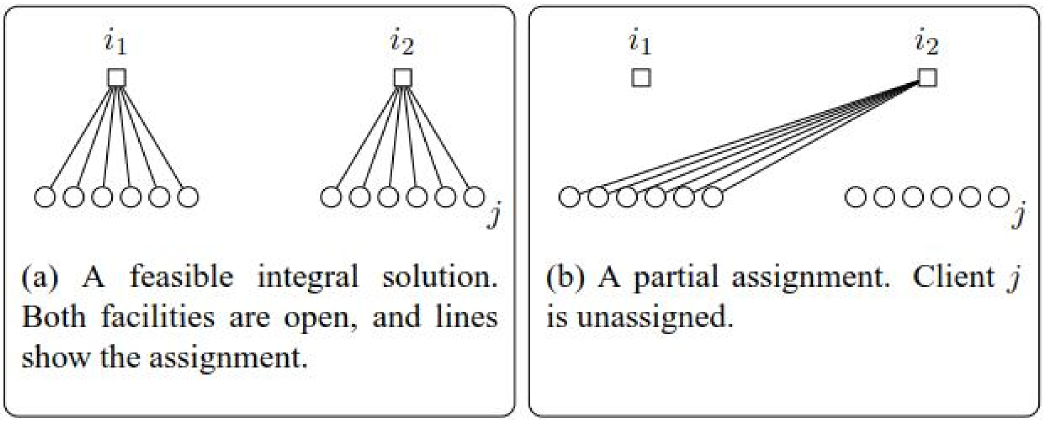
- Residual instance specifics:

  $\forall i \in \mathcal{F}: \quad \bar{u}_i = u_i - g^{-1}(i)$ ← # of clients assigned to i in g

  Clients: $\quad \bar{\mathcal{D}} = \mathcal{D} \setminus \mathcal{D}'$

• Thought: integral soln $(x, y)$ should be feasible for residual instance for feasible partial assignment

Wrong:




(a) A feasible integral solution. Both facilities are open, and lines show the assignment.

(b) A partial assignment. Client $j$ is unassigned.

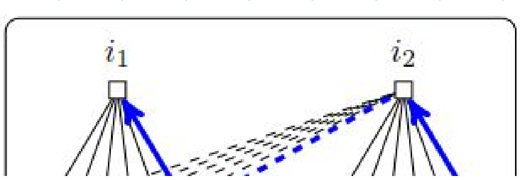
Williamson, Shmoys Open Problem 5

Is there a relaxation-relative $O(1)$-apx for CFL?

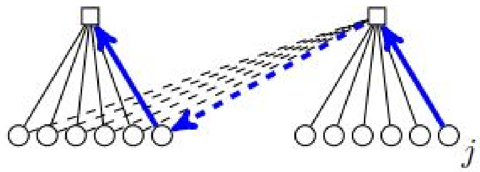
Thm [An, Singh, Svensson '14] Yes!

Insights from above example: Client $j \notin D'$ can't take its normal spot as some other client $j' \in D'$ has take it.

Then $j'$ must free up some space elsewhere.




Want to allow the possibility to undo partial assignment.



(c) Original solution, augmented with partial assignment edges marked as dashed lines. Thick lines represent alternating path for $j$.

to undo partial assignment.

Think: alternating path

LP uses flow-cover like constraints and we do not know how to solve LP efficiently.

[ Open: find an efficiently solvable, high quality relaxation for CFL ]

[ASS14] propose relaxed separation oracle
(recall: cap. set cover discussion)

Rough plan

① Give fraction LP soln in the natural space $(x, y)$

Open all facilities that are open to constant extent:

$$I = \{ i \in F : y_i \geq \frac{1}{\gamma} \}$$

$\gamma = O(1)$

② Find a max-size assignment of clients to $I$ that is not too expensive.
(i.e.: assignment cost bd. by $2c^T x$)

(2) Done if all clients can be assigned

Oth: use partial assignment to formulate multicommodity flow prob.

- one commodity for each unassigned client (say $D'$)
- client $j$ needs to sink a unit value at some facility with pos. remaining capacity

- facilities in $F \setminus I$ have residual cap $y_i \cdot u_i$

Key trick show that we can find such a feasible flow that sinks half of each commodity's demand in $F \setminus I$ (fac. $i$ with small $y_i$), assigning $\le y_i u_i$ demand to each $i \in F \setminus I$.

Gives rise to residual fac loc instance with clients $D'$ and facilities $F \setminus I$ and soln $(\bar{x}, \bar{y})$ s.t.

$$\sum_{i \in F \setminus I} \bar{x}_{ij} \geq \frac{1}{2} \quad \forall j \in D'$$

and $$\sum_{j \in D'} \bar{x}_{ij} \leq \bar{y}_i u_i < \frac{1}{\alpha} u_i \quad \forall i \in F \setminus I$$

**Now:** use apx for soft-cap FL
[Abrams, Meyerson, Munagala, Plotkin '02]

Given $(2\bar{x}, 2\bar{y})$ find integral solution $(\hat{x}, \hat{y})$

① with cost
$\leq 18 \cdot (2c^T\bar{x} + 2f^T\bar{y})$

② at most $2 \cdot (2\bar{y}_i \cdot u_i) < \frac{4}{\gamma} u_i$
clients assigned to $i \in F \setminus I$

Choose $\gamma \geq 4$ to get feasibility.

[ **Open:** Improve (18,2)-apx of Abrams et al ]

**Also:** [Bansal, Garg, Gupta '12]
5-apx for CFL via local search

[ **Open:** Can we use this somehow to improve rounding of ASS'14 LP? ]

**Details** Given a valid partial assignment $\{g_{ij}\}_{i \in F, j \in D}$ (0,1)

$\forall j \in D: \sum_{i \in F} g_{ij} \leq 1$

$\forall i \in F \quad \sum_{j \in D} g_{ij} \leq u_i$

$$D' = \{ j \in D : \textstyle\sum_i g_{ij} = 0 \}$$

## Multi-Commodity Flow network   MFN $(g, x, y)$

Fix digraph $\mathcal{H}$ with

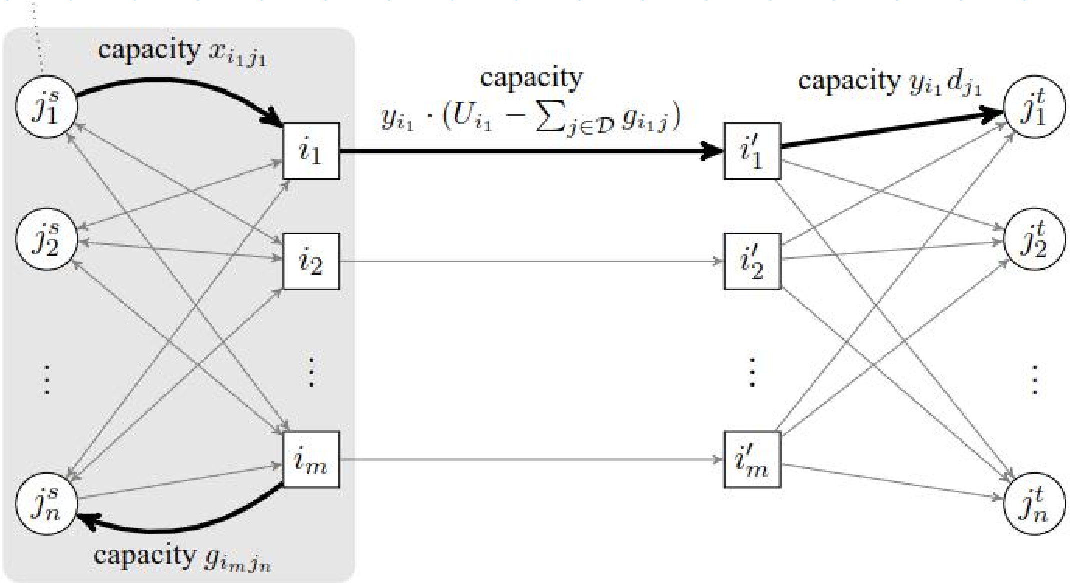
- nodes $\mathcal{N} = \{ j_s, j_t : j \in D \} \cup \{ i, i' : i \in \mathcal{F} \}$
- arcs $A = \{ (i, i') : i \in \mathcal{F} \} \cup \{ (i', j_t) : i \in \mathcal{F}, j \in D \} \cup \{ (j_s, i), (i, j_s) : i \in \mathcal{F}, j \in D \}$
- Commodity with demand 1 between $j_s, j_t$ $\forall j \in D'$.
- capacities :

  Commodity oblivious capacity of $i$

  - $(i, i')$ : $y_i (u_i - \sum_j g_{ij})$   $\forall i \in \mathcal{F}$
  - $(j_s, i)$ : $x_{ij}$   $\forall i \in \mathcal{F}, j \in D$
  - $(i, j_s)$ : $g_{ij}$   $\forall i \in \mathcal{F}, j \in D$
  - $(i', j_t)$ : $y_i d_j$   $\forall j \in D'$, $0$ other.

$$\Big( d_j = 1 - \underbrace{\sum_{i \in \mathcal{F}} g_{ij}}_{0,1 \text{ for us}} \Big)$$

Commodity-specific cap of $i$




## Strengthened LP

$\overline{P}$ $\quad \min \; c(x,y) = \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$

s.t. $\quad MFN(g,x,y)$ feasible for all valid partial flows $g$

$x \in [0,1]^{\mathcal{F} \times \mathcal{D}}, \; y \in [0,1]^{\mathcal{F}}$

Recall: Do not know how to solve $\overline{P}$ !

## Observations

① $g = 0$ is valid, $MFN(0,x,y)$

Is equivalent to standard CFL LP.
Let $f$ be a flow for $MFN(0,x,y)$.

Decompose $f$ into paths $\mathcal{P}$ (and cycles).

Def:

$$x_{ij} = \sum_{\substack{P \in \mathcal{P} \\ j_s, j_t\text{-path} \\ \text{using } (i,i')}} f_P \qquad \forall i,j$$

$$y_i = \sum_{\substack{P \in \mathcal{P} \\ \text{using } (i,i')}} f_P \qquad \forall i$$

Check: $(x,y)$ is feasible for standard LP

(2) Suppose $(x,y)$ is feasible for $(\bar{P})$ with integral valid partial flows

$\Longrightarrow$ also feas. for $(\bar{P})$ with fractional valid partial flows

(fractional partial flows are convex comb. of integral ones

$\longrightarrow$ Lemma 2.3)

(3) Given $(x^*, y^*)$ and valid int. partial flow $g^*$ we can check efficiently whether $MFN(g^*, x^*, y^*)$ is feasible. If infeasible, can efficiently obtain violated ineq. in $x,y$-space.

$\Longrightarrow$ need this for relaxed sep oracle

(4) $(\bar{P})$ is a relaxation of CFL

relatively straight forward
$\longrightarrow$ Sec 2.2 of paper

## Algorithm Overview (Rough)

- Guess cost opt of an optimal solution to CFL instance, and add

$$c^T x + f^T y \leq \text{opt} \qquad (*)$$

to standard CFL LP.

- Solve LP via Ellipsoid
Given candidate $(x, y)$ first verify $(*)$ and box constraints.

$(*)$ [ Carefully choose a certain specific valid partial assignment $g$.

If $MFN(g, x, y)$ feasible $\Rightarrow$ can round $x, y$ to an $O(1)$-apx CFL soln.

otherw. generate viol. ineq. $\rightarrow$ continue.

## $(*)$ Finding $g$

Let $I = \{ i : y_i \geq \frac{1}{2} \}$ $\quad S = \mathcal{F} \setminus I$

(W)

**Want** find valid partial (int.) assignment $g$ some demands into $I$

(a) that is cheap : $c^T g + f(I) \le opt$

(b) $D' = \{ j \in D : \sum_i g_{ij} = 0 \}$

$\exists$ feasible flow $f$ for $MFN(g, x, y)$

that drains $D'$-commodities in $S$

**Why is (b) useful?**

$\Longrightarrow$ $D'$ is served entirely by $S$

and $\underbrace{g \text{ maps } D \setminus D' \text{ into } I}_{\text{indep. of flows}}$

**Final soln** : • $j \in D \setminus D'$ is assigned to $i \in I$ if $g_{ij} = 1$

• use feasible flow for $MFN(g, x, y)$ to assign $D'$ to $S$

Let $f$ be such a flow with flow decomp $\mathcal{P} \cup \mathcal{C}$

paths, cycles

$$x_{ij} = \sum_{\substack{P \in \mathcal{P}_{j, s, j_t} \\ (i, i') \in P}} f_P$$

$$y_i = \sum_{P : (i, i') \in P} f_P$$

**note:** $\sum_{i \in S} x_{ij} = 1 \quad \forall j \in D'$

<u>note:</u>

$P_{res}$

$$\sum_{i \in S} x_{ij} = 1 \quad \forall j \in D'$$

$$x_{ij} \le y_i \quad \forall i \in S, j \in D' \qquad (y_i < 1/2)$$

$$\sum_{j \in D'} x_{ij} \le y_i \cdot u_i \quad \forall i \in S$$

$$0 \le x, y \le \mathbb{1}$$

Standard LP for CFL inst. ind. by snd inst.

Use Abrams et al. (18,2)-apx to round $(x,y)$ into integral $(\bar{x},\bar{y})$ s.t. — SOFT CFL

$$\sum_i \bar{x}_{ij} = 1 \quad \forall j \in D'$$

$$\sum_{j \in D \setminus D'} \bar{x}_{ij} \le 2 \cdot y_i u_i \boxed{\le u_i}$$

$$\Rightarrow \quad c^T \bar{x} + f^T \bar{y} \le 18 \cdot (c^T x + f^T y)$$

Adding assignment costs of $g$ gives another add. opt. in cost. Note

<u>Determine $g$</u>

For simplicity assume
$$c_{ij} = c_{i'j} \quad \forall i,i' \in F, j \in D \qquad (A)$$

Given $(x,y)$ let $I = \{ i : y_i \ge \frac{1}{2} \}$.

Set up bip. $b$-matching instance with

- nodes $D \cup I$, all edges $E = \{ij : i \in I, j \in D, x_{ij} > 0\}$
- $b_i = u_i \quad \forall i \in I$
- $b_j = 1 \quad \forall j \in D$

$g$: max $b$-matching in above graph

<u>note:</u> $\sum_i c_{ij} g_{ij} = c_j \sum_i g_{ij}$

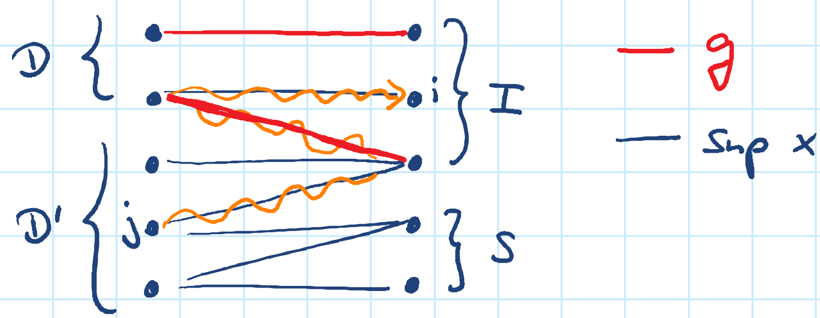
$$= c_j \underbrace{\sum_i x_{ij}}_{=1}$$

$\forall$ matched $j$.

$\Rightarrow g$ is cheap

<u>Claim:</u> $(x,y)$ feasible for $(\bar{P}) \Rightarrow MFN(g,x,y)$ has feasible $f$ that drains all flow in $S$

<u>Pf:</u> (sketch) Suppose not true, every feasible flow drains some amount in $I$




There is no way to assign $D'$ to $S$.
In feasible solns $f$ to $MFN(g,x,y)$ some flow sinks in $I$.

$\Longrightarrow$ Gives rise to augmenting path in b-matching. ▨

MFN$(g, x, s)$ having feas flow that obtains in S

|||

$(P_{res})$ being feasible

If $(P_{res})$ not feasible $\Longrightarrow$ add violated inequ. to LP

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Everything also works if $c_{ij} \neq c_{i'j}$ for some $j, i, i'$.
Need to construct fractional flow g. Arguing that cheapones can be found is harder.

$\Longrightarrow$ 288-apx instead of 18

Improve this!