Intro: Steiner Trees via Graph Decomposition

Tuesday, April 5, 2016 2:49 PM

Given: minimization problem $\mathcal{P}$

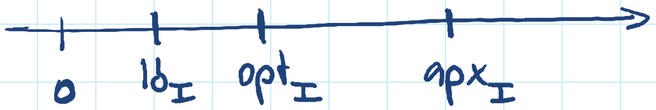
An algo APX is called an α-approximation if given an instance I of $\mathcal{P}$

ⓐ APX produces in poly time a feasible soln APX_I, and

ⓑ The obj val. apx_I is at most $\alpha \cdot opt_I$

↑ obj. val. of OPT_I

Challenge: $\mathcal{P}$ is often NP-hard and hence we don't usually know opt_I



Find a polynomial-time computable lowerbound $lb_I \le opt_I$ s.t.

$$\frac{apx_I}{opt_I} \le \frac{apx_I}{lb_I} \le \alpha$$

Popular approach use mathematical programming

① lb_I is the optimal value of an LP/SDP/Convex NLP relaxation of the prob.

② optimal solutions to the relaxations
can efficiently be rounded
into "high-quality" integral solutions

This class Approx is an exciting vibrant area.
Give some recent examples to convince you.
[Focus on ①,②]

Most examples will be recent but some
may also be (elegant) novel proofs of older
results

Goals ① Add versatile techniques to your toolkit

② Interest you in area (by pointing out open
problems)

Oral exam July 20

Prerequisites Graph algorithms, linear & combinatorial
optimization, basics of Approx
→ KV

Lecture notes: will post handwritten notes

Rough Topic list (i) Intro : Approximate Network Design
(direct LP rounding, graph decomposition,
iterative techniques)

(ii) Strengthening LPs

adding strong valid inequalities,
lift-and-project (Rothvoss)
rounding solutions to such LPs

(iii) Use Discrepancy theory to round fractional solns to LP

(Lovett, Meka '12; Bansal, Nagarajan '16)

Intro Steiner tree problem

input: undirected graph $G = (V, E)$, set of terminals $R \subseteq V$, edge costs $c_e \geq 0 \ \forall e \in E$.

goal: compute minimum cost tree T spanning R

natural IP

$$\text{(IP)} \quad \min \overbrace{\sum_{e \in E} c_e x_e}^{c^T x}$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall \overbrace{S \subseteq V \text{ s.t. } S \cap R \neq \emptyset, \ V \setminus S \cap R \neq \emptyset}^{\text{Steiner cuts}}$$

$x(\delta(S)) \longrightarrow \sum_{e \in \delta(S)}$

$$x_e \in \{0,1\} \quad \forall e$$

(validity: x feasible, $\operatorname{supp}(x) = \{e \in E : x_e = 1\}$

Claim: $\operatorname{supp}(x)$ has u,v-path $\forall u,v \in R$)

(ex)

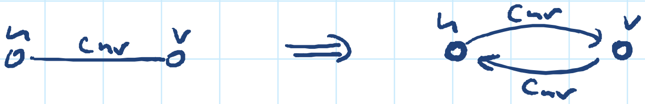
Thm 1 There is a 2-apx for Steiner tree

Pf: Compute an extreme point solution x^* of LP relaxation Ⓟ of (IP). How?

x^* is rational (→ KV chp 3)

Let M s.t. Mx^* is integral. Create multigraph on vertex set V by inserting Mx^*_e copies of each edge e.

Form digraph D by replacing each edge



Pick $r \in R$ arbitrarily.

Let $\overbrace{\lambda_D(r,v)}^{\lambda_v}$ be the # arc-disjoint dipaths in D, let $d^-(v)$ & $d^+(v)$ be in and out degree.

note: $\lambda_v \geq M \quad \forall v \in R$, $d^-(v) = d^+(v) \quad \forall v \in V$

Powerful tool:

Thm 2 (Bang-Jensen, Frank & Jackson '95)

$\geq$ suffices

$D = (N, A)$, $d^-(v) \boxed{=} d^+(v) \quad \forall v \in N \setminus r$

λ_v: # arc disj. r,v-dipaths

$\Longrightarrow \exists$ arc disj. $A_1, \dots A_p$ arborescences s.t.

$\forall v \in N \setminus r$: v is in $\geq \lambda_v$ of the arbs

$(p = \max_v \lambda_v)$.

$\forall v \in V \setminus R$: v is in $\geq \lambda_v$ of the arbs

($p = \max_v \lambda_v$)

Thm 2 applies here and yields arc disjoint arborescences

$A_1, \dots, A_p$ ($p = \max_{v \in V} \lambda_v$

each spanning R! $= M$)

↑ why?

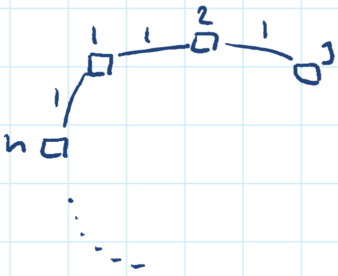
Let χ_i be the characteristic vector of A_i. Then

(repl. each edge by 2 arcs)

$$\frac{1}{M} \sum_{i=1}^{M} \chi_i \leq 2x^*$$

$$\Longrightarrow \min_i c^T \chi_i \leq 2c^T x^* \leq 2 \cdot \text{opt} \quad ▨$$

Can we do better with this LP? No!



$R = V = \{1..n\}$

$c_{12} = c_{23} = \dots = c_{n1} = 1$

$\text{opt}_{IP} = n-1$

$\text{opt}_{P} = n/2$

Later: pf of Thm 2 & stronger LPs for Steiner trees

First: using graph decomposition

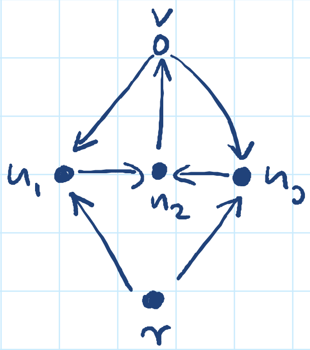
In Thm 2, can replace $d^-(v) = d^+(v)$ by $d^-(v) \geq d^+(v)$.

Idea: create new digraph D' by adding $d^-(v) - d^+(v)$ copies of arc (v,r) to D, for all $v \in N \setminus r$. ↑ auxiliary

$\Rightarrow$ D' is Eulerian and applying Thm 2 to D' yields arborescences that do not use auxiliary arcs.

Why is $d^-(v) \geq d^+(v)$ condition important?

Eg [Lovász]



$\exists$ 2 arc-disj r,u_i-paths $\forall i=1,2,3$

There are no 2 arc-disj. arb. that both span r,u_1,u_2,u_3.

note: $d^-(v) < d^+(v)$!

Thm 2b [BFJ '95]

$D=(N,A)$, $r \in N$ $\quad T' = \{v \in N \setminus r : d^-(v) < d^+(v)\}$

If $\lambda_D(r,v) \geq k$ $\forall v \in T'$ then there are arc disj. arb. $A_1, \dots, A_k$ that each $v \in N$ belongs to $\min\{k, \lambda_D(r,v)\}$ of these.

Examples

Examples

① mincost arborescences $D(N,A)$ $r \in N$ $c_a \geq 0$ $\forall a \in A$

goal: find arb. T rooted at r, spanning N

(P_1) min $\sum_a c_a x_a$

s.t.

$$\sum_{a \in \delta^+(S)} x_a \geq 1 \quad \forall\, S \subsetneq N, r \in S$$

$$x \geq 0$$

Thm 3 (Edmonds '67) (P_1) is integral

Pf: Let x be an extreme pt soln to (P_1) and M s.t. $\bar{x} = Mx$ is integral.

Thm 26 $\Rightarrow$ $\exists$ arb. $A_1, \ldots, A_M$ that span N s.t.

$$x \geq \frac{1}{M} \sum_{i=1}^{M} \chi(A_i)$$

↑ feasible f. (P_1)

$\Rightarrow$ $c(A_i) \leq c^T x$ for some i

$\Rightarrow$ (P_1) has an optimal integral soln for all $c \geq 0$.

$\Rightarrow$ (P_1) is integral ▨

↑ [Edmonds & Giles '77] see Thm 5.13 VV

② Price collecting Steiner trees $G = (V,E)$, $c_e \geq 0$ $\forall e \in E$

② Price collecting Steiner trees (PCST) $G=(V,E)$, $c_e \geq 0 \ \forall e \in E$

penalty $\pi_v \geq 0 \ \forall v \in V$, $r \in V$

goal: Find tree T rooted at r s.t.

$$\sum_{e \in E(T)} c_e + \sum_{v \notin V(T)} \pi_v$$

is minimized

$x_e = 1$ if $e \in E(T)$

$$\boxed{P_2} \quad \min \sum_e c_e x_e + \sum_v \pi_v z_v \longleftarrow z_v = 1 \text{ if } v \notin V(T)$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e + z_v \geq 1 \quad \forall S \subseteq V, r \in S, v \notin S$$

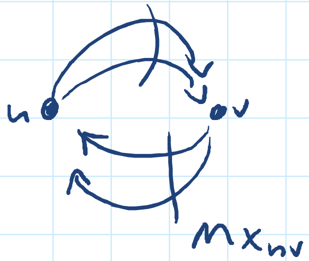
$$x, z \geq 0$$

$\forall v \notin S$, $\delta(S)$ must contain e of T or v is not spanned by T

Solve $(P_2) \Longrightarrow (x,z)$

$\bar{x}, \bar{z} = M(x,z)$ integral for some M

$D=(V,A)$ where A has Mx_{uv} copies of (u,v) and (v,u) $\forall uv \in E$.



note: min r,v-cut in D has $\geq (1-z_v) \cdot M$ arcs

max flow-min cut $\Longrightarrow \lambda_D(r,v) \geq (1-z)M$

Thm 2 $\Longrightarrow$ $\exists$ M arb. $A_1, \dots, A_M$ s.t.

$$\frac{1}{M} \sum_i x(A_i) \le 2x \quad \text{s.t.}$$

$$|\{ i : v \in V(A_i) \}| \ge \lambda_D(r,v) \ge (1-z)M$$

$$\forall v \in V$$

Let $\bar{A}$ be random arb. from $\{A_1, \dots, A_M\}$.
Then

$$Pr[v \notin V(\bar{A})] \le z_v$$

$$Pr[e \in E(\bar{A})] \le 2x_e$$

$$\Longrightarrow \quad E[\sum_{e \in \bar{A}} c_e + 2 \sum_{v \notin V(\bar{A})} z_v] \le 2(c^T x + \pi^T z)$$

Theorem 4 (Goemans & Williamson '95)

There is $\underbrace{\text{(Lagrangean-Multiplier Preserving)}}_{\text{LMP}}$

2-apx for PCST.

Notes: LMP property can be used to...

① approximate partial Steiner tree:
find mincost StT spanning $\ge K$ terminals

[Chudak, Roughgarden, Williamson '04]

② get a $(2-\epsilon)$-apx for PCST ($\epsilon > 0$ small)
[Archer et al. '11]