Exercise Set 9

Exercise 9.1:
Prove Kraft’s inequality (Proposition 4.1 in the lecture notes).
Let $S$ be a nonempty finite set, $r \notin S$, and $h_s \in \mathbb{N}$ for $s \in S$. There exists a topology for root $r$ and sinks $S$ with $|E(A_{[r,s]})| - 1 \leq h_s$ for all $s \in S$ if and only if
\[
\sum_{s \in S} 2^{-h_s} \leq 1.
\]
(5 points)

Exercise 9.2:
Consider the Repeater Tree Topology problem. Let $|S| = n \geq 2$, and for each sink $s_i$ define $a'_{s_i} := a_{s_i} - d\|p(r) - p(s_i)\|_1, i = 1, \ldots, n$.

Algorithm 1: Huffman coding
\[
A \leftarrow (\{r\}, \emptyset);
\]
for $k = n, n-1, \ldots, 2$ do
  Sort $S = \{s_1, \ldots, s_k\}$ s.t. $a'_{s_1} \leq \cdots \leq a'_{s_k}$;
  Merge two sinks $s_{k-1}, s_k$ into a new sink $s'_{k-1}$;
  Choose $p(s'_{k-1})$ as the position that minimizes
  \[
  \|p(r) - p(s'_{k-1})\|_1 + \|p(s_k) - p(s'_{k-1})\|_1 + \|p(s_{k-1}) - p(s'_{k-1})\|_1;
  \]
  Set $a_{s'_{k-1}} := \min\{a_{s_i} - d\|p(s_i) - p(s'_{k-1})\|_1 - b \mid i = k, k-1\};$
  /* Update $S$, add vertices and edges to $A$ */
  $S \leftarrow (S \setminus \{s_{k-1}, s_k\}) \cup \{s'_{k-1}\};$
  $V(A) \leftarrow V(A) \cup \{s_{k-1}, s_k, s'_{k-1}\};$
  $E(A) \leftarrow E(A) \cup \{(s'_{k-1}, s_{k-1}), (s'_{k-1}, s_k)\};$
  /* Connect the last vertex to $r$ */
  $E(A) \leftarrow E(A) \cup \{(r, s'_{1})\}$.

Show that the topology generated by Algorithm 1 maximizes the worst slack.
(5 points)

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Exercise 9.3:
Show that the approximation algorithm for the Rectilinear Sink Clustering Problem presented in the lecture can be implemented to run in $O(|D| \log |D|)$ time.

Note: you can assume that a shortest rectilinear spanning tree on $n$ terminals can be computed in $O(n \log n)$ time.

(5 points)