Research Institute for Discrete Mathematics
Approximation Algorithms
Summer Term 2016

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Exercise Sheet 5

Problem 5.1. (2 points)
The Knapsack Problem can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \leq W,\, \forall\, 1 \leq i \leq n : x_i \in \{0,1\} \right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LR}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0,1\}$ is replaced by $0 \leq x_i \leq 1$.

Show that the *integrality gap*

$$\sup_{\mathcal{I}} \left\{ \frac{\mathrm{LR}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \,:\, \mathrm{OPT}(\mathcal{I}) \neq 0 \right\}$$

of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with $w_i \leq W$ for all $i = 1, \ldots, n$?

Problem 5.2. (5 points)
Show that the Knapsack Problem given in exercise 5.1 remains NP-complete if we require $x_i \in \mathbb{Z}_{\geq 0}$ instead of $x_i \in \{0,1\}$ for $i = 1, \ldots, n$.

Problem 5.3. (4 points)
Describe a polynomial-time combinatorial algorithm for the Fractional Multi Knapsack Problem: Given natural numbers $n, m \in \mathbb{N}$ and w_i, c_{ij} as well as W_j for $1 \leq i \leq n$ and $1 \leq j \leq m$, find x_{ij} satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} x_{ij} w_i \leq W_j$ for all $1 \leq j \leq m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum.

Problem 5.4. (5 points)
Describe exact algorithms with running time $\mathcal{O}\left(2^{n/2}\right)$ for the following problems:

(i) Subset Sum: Given $K, n, x_1, \ldots, x_n \in \mathbb{N}$, find $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = K$.

(ii) Knapsack Problem (where n denotes the number of items).

Please hand in your solutions on Tuesday, **May 24th**, before the lecture.