Summer Term 2015
Prof. Dr. Stefan Hougardy
Anna Hermann

Research Institute for Discrete Mathematics
University of Bonn

Approximation Algorithms

Exercise Sheet 5

Exercise 5.1:
The Knapsack Problem can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \; : \; \sum_{i=1}^{n} w_i x_i \le W,\, x_i \in \{0,1\} \, \forall \, 1 \le i \le n \right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LR}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0,1\}$ is replaced by $0 \le x_i \le 1$.
Show that the *integrality gap*

$$\sup_{\mathcal{I}} \left\{ \frac{\mathrm{LR}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \; : \; \mathrm{OPT}(\mathcal{I}) \neq 0 \right\}$$

of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with $w_i \le W$ for all $i = 1, \dots, n$?

(2 points)

Exercise 5.2:
Describe a $\frac{3}{4}$-approximation algorithm for the Knapsack Problem with running time $\mathcal{O}(n^3)$ and give a proof of the approximation ratio.

Note: You may not use the FPTAS for Knapsack.

Hint: The basic idea is to run, for every pair of items, a $\frac{1}{2}$-approximation algorithm on a subset of the remaining elements.

(5 points)

Exercise 5.3:
Consider the Fractional Multi Knapsack Problem: Given natural numbers $n, m \in \mathbb{N}$ and weights w_i, W_j as well as costs c_{ij} for $1 \le i \le n$ and $1 \le j \le m$, find $x_{ij} \in [0,1]$ satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \le i \le n$ and $\sum_{i=1}^{n} x_{ij} w_i \le W_j$ for all $1 \le j \le m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum.
State a polynomial-time combinatorial algorithm for this problem.

(4 points)

Exercise 5.4:
Let $\mathcal{P}$ be an optimization problem such that any instance I of $\mathcal{P}$ consists of a list of non-negative integers, with $\text{largest}(I)$ the largest of these integers. We call $\mathcal{P}$ *strongly NP-hard* if there is a polynomial p such that the restriction of $\mathcal{P}$ to instances I satisfying $\text{largest}(I) \leq p(\text{size}(I))$ is NP-hard.

(i) Show that if $\mathcal{P}$ is strongly NP-hard with integral objective function satisfying

$$\text{OPT}(I) \leq q(\text{size}(I), \text{largest}(I))$$

for some polynomial q and all instances I, then $\mathcal{P}$ has no fully polynomial approximation scheme unless $P = NP$.

Consider the following generalization of the Knapsack Problem: Given $n \in \mathbb{N}$ items with weight $w_j \in \mathbb{N}$ for $1 \leq j \leq n$ together with $W \in \mathbb{N}$ and costs $c_{ij} \in \mathbb{N}$ for $i, j \in \mathbb{N}$, $1 \leq i < j \leq n$, determine a subset $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} w_i \leq W$ such that the overall cost $\sum_{i,j \in S:\, i<j} c_{ij}$ is maximum.

(ii) Show that (provided $P \neq NP$) no fully polynomial approximation scheme for this problem exists.

(2 + 3 points)

Please turn in your solutions on Tuesday, **May 19th**, before the lecture.