Approximation Algorithms

Exercise Sheet 5

Exercise 5.1:
The Knapsack Problem can be formulated as integer program:

\[
\max \left\{ \sum_{i=1}^{n} c_i x_i : \sum_{i=1}^{n} w_i x_i \leq W, \; x_i \in \{0,1\} \; \forall \; 1 \leq i \leq n \right\} \tag{1}
\]

For an instance \( I \), denote the optimum of (1) by OPT(\( I \)) and let LR(\( I \)) be the optimum of the linear relaxation, where \( x_i \in \{0,1\} \) is replaced by \( 0 \leq x_i \leq 1 \).

Show that the integrality gap

\[
\sup_{I} \left\{ \frac{\text{LR}(I)}{\text{OPT}(I)} : \text{OPT}(I) \neq 0 \right\}
\]

of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with \( w_i \leq W \) for all \( i = 1, \ldots, n \)?

(2 points)

Exercise 5.2:
Describe a \( \frac{3}{4} \)-approximation algorithm for the Knapsack Problem with running time \( O(n^3) \) and give a proof of the approximation ratio.

Note: You may not use the FPTAS for Knapsack.

Hint: The basic idea is to run, for every pair of items, a \( \frac{1}{2} \)-approximation algorithm on a subset of the remaining elements.

(5 points)

Exercise 5.3:
Consider the Fractional Multi Knapsack Problem: Given natural numbers \( n, m \in \mathbb{N} \) and weights \( w_i, W_j \) as well as costs \( c_{ij} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), find \( x_{ij} \in [0,1] \) satisfying \( \sum_{j=1}^{m} x_{ij} = 1 \) for all \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} x_{ij} w_i \leq W_j \) for all \( 1 \leq j \leq m \) such that \( \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij} \) is minimum.

State a polynomial-time combinatorial algorithm for this problem.

(4 points)
Exercise 5.4:
Let $\mathcal{P}$ be an optimization problem such that any instance $I$ of $\mathcal{P}$ consists of a list of non-negative integers, with $\text{largest}(I)$ the largest of these integers. We call $\mathcal{P}$ strongly $NP$-hard if there is a polynomial $p$ such that the restriction of $\mathcal{P}$ to instances $I$ satisfying $\text{largest}(I) \leq p(\text{size}(I))$ is $NP$-hard.

(i) Show that if $\mathcal{P}$ is strongly $NP$-hard with integral objective function satisfying

$$\text{OPT}(I) \leq q(\text{size}(I), \text{largest}(I))$$

for some polynomial $q$ and all instances $I$, then $\mathcal{P}$ has no fully polynomial approximation scheme unless $P = NP$.

Consider the following generalization of the Knapsack Problem: Given $n \in \mathbb{N}$ items with weight $w_j \in \mathbb{N}$ for $1 \leq j \leq n$ together with $W \in \mathbb{N}$ and costs $c_{ij} \in \mathbb{N}$ for $i, j \in \mathbb{N}$, $1 \leq i < j \leq n$, determine a subset $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} w_i \leq W$ such that the overall cost $\sum_{i,j \in S; i < j} c_{ij}$ is maximum.

(ii) Show that (provided $P \neq NP$) no fully polynomial approximation scheme for this problem exists.

(2 + 3 points)

Please turn in your solutions on Tuesday, May 19th, before the lecture.