Exercise Set 10

Exercise 10.1:
Prove: For $d = 2$, the optimum objective function value of the spreading LP is a lower bound for the optimum objective function value of the corresponding instance of the 2-DIMENSIONAL ARRANGEMENT PROBLEM.

(4 points)

Exercise 10.2:
Consider the special case of the QUADRATIC ASSIGNMENT PROBLEM where $|U| = |V(G)|$, $w(e) = 1$ for all $e \in E(G)$, $d$ is metric, $c$ is zero, and $G$ is a wheel, i.e. for even $n$ we have $V(G) = \{v_1, ..., v_n\}$ and $E(G) = E_1 \cup E_2$ with $E_1 = \{\{v_i, v_{i+1}\} : i = 1, ..., n\}$ and $E_2 = \{\{v_i, v_{i+\frac{n}{2}}\} : i = 1, ..., \frac{n}{2}\}$, where all indices are modulo $n$. Let $f^*$ be the embedding such that $\{\{f^*(x), f^*(y)\} : \{x, y\} \in E_1\}$ is a shortest TSP tour on $U$ with respect to $d$.

(a) Show that $\sum_{e=\{x,y\} \in E(G)} d(f^*(x), f^*(y)) = \Omega(n \cdot OPT)$, where $OPT$ denotes the optimum objective function value of the given instance of the QUADRATIC ASSIGNMENT PROBLEM.

(b) Give a polynomial time 3-approximation algorithm for the above special case of the QUADRATIC ASSIGNMENT PROBLEM.

(4 + 4 points)

Exercise 10.3:
Let $G = (V, E)$ be a simple undirected graph with $V = \{1, ..., n\}$. The Laplacian matrix $L_G$ of $G$ is the $n \times n$-matrix whose entries $l_{i,j}$, $1 \leq i, j \leq n$, are given by

\[
l_{i,j} = \begin{cases} 
-1 & \text{if } \{i, j\} \in E, \\
|\delta(i)| & \text{if } i = j, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Prove that $L_G$ is positive semidefinite, that is, $x^T L_G x \geq 0$ for all $x \in \mathbb{R}^n$.

(b) Let $G$ be connected and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $L_G$. Show that $\lambda_1 = 0$ and $\lambda_2 > 0$.

(c) Show that the multiplicity of 0 as an eigenvalue of $L_G$ equals the number of connected components of $G$.

(1 + 1 + 2 points)
Deadline: Thursday, June 26, before the lecture.  
The websites for lecture and exercises are linked at  
http://www.or.uni-bonn.de/lectures/ss14/ss14.html  
In case of any questions feel free to contact me at scheifele@or.uni-bonn.de.  

Note that this will be the last exercise sheet (except for programming exercises) that will be relevant for admittance to the exam.