Exercise Set 4

Exercise 4.1:
A Steiner topology is called full if all terminals have degree one. Let $f(k)$ denote the number of full topologies for the Rectilinear Steiner tree problem with $k$ terminals in which all Steiner points have degree 3. Derive and prove a formula on $f(k)$.

(4 points)

Exercise 4.2:
Let $F$ be a shortest rectilinear Steiner tree for a set $Z \subset \mathbb{R}^2$ of terminals. For $u, v \in V(F)$ define

\[
\mathcal{L}(u, v) := \{p \in \mathbb{R}^2 : ||p - u||_1 < ||u - v||_1 \text{ and } ||p - v||_1 < ||u - v||_1\}
\]
\[
\mathcal{R}(u, v) := \{p \in \mathbb{R}^2 : ||u - v||_1 = ||u - p||_1 + ||p - v||_1\}
\]

Prove:

(a) If \{u, v\} $\in E(F)$, then $\mathcal{L}(u, v)$ contains no terminal, Steiner point or interior segment point of $F$.

(b) If $u, v, w \in E(F)$ such that \{u, w\} and \{w, v\} are perpendicular segments of $F$, then $\mathcal{R}(u, v)$ contains no terminal, Steiner point or interior segment point of $F$.

(2+2 points)

Definition: Let $T \subset \mathbb{R}^2$ be a finite set and let $r \in \mathbb{R}^2$.

- A rectilinear shortest path tree for $T + r$ is a rectilinear Steiner tree $F$ for instance $T \cup \{r\}$ such that the $r$-$t$ path contained in $F$ is a shortest path w.r.t. $L_1$ distances for all $t \in T$.
- A minimum cost rectilinear shortest path tree for $T + r$ is a rectilinear shortest path tree $F$ for which $c(F) := \sum_{(v,w) \in E(F)} ||v - w||_1$ is minimum.
- We denote the cost of a minimum cost rectilinear shortest path tree by $\text{rspt}(T + r)$.
- For $p_1, p_2, p_3 \in \mathbb{R}^2$ we define $\text{med}(p_1, p_2, p_3) \in \mathbb{R}^2$ to be the point with $x$- (resp. $y$) coordinate equal to the median of the $x$- (resp. $y$) coordinates of $p_1, p_2$ and $p_3$. 

Exercise 4.3:
Consider the following algorithm:

\[
F := (T \cup \{r\}, \emptyset)
\]

while \(|T| > 1\) do

choose \(t_1, t_2 \in T\) maximizing \(||\text{med}(r, t_1, t_2) - r||_1\)

include \(\text{med}(r, t_1, t_2)\) to \(V(F)\)

connect \(\text{med}(r, t_1, t_2)\) with \(t_1\) and \(t_2\) in \(F\)

set \(T = (T \setminus \{t_1, t_2\}) \cup \{\text{med}(r, t_1, t_2)\}\)

include an edge between \(r\) and the remaining element of \(T\) to \(E(F)\)

\text{return } F

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Example of the algorithm. Root \(r\) is plotted in green. The blue vertices depict \(T\).

Let \(F\) be the output of the algorithm. Prove:

(a) \(F\) is a rectilinear shortest path Steiner tree.

(b) \(c(F) \leq 2 \cdot \text{rspt}(T + r)\) if \(r = (0, 0)\) and all terminals \(t \in T\) are located in the first quadrant.

(Remark: The result holds for general instances)

(c) The approximation ratio of 2 is tight.

(d) The algorithm can be implemented to run in \(O(|T| \cdot \log(|T|))\) time.

\((2+6+4+4\text{ points})\)

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Deadline: Thursday, May 16th, before the lecture.