Research Institute for Discrete Mathematics
Approximation Algorithms
Summer Term 2013

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Exercise Sheet 3

Exercise 3.1:
An instance of MAXIMUM SATISFIABILITY is called k-satisfiable if any k of its clauses can be satisfied simultaneously. Let r_k be the infimum of the fraction of clauses that one can satisfy in any k-satisfiable instance. Show:

1. $r_1 = \frac{1}{2}$.
2. $r_2 = (\sqrt{5} - 1)/2$.

   Hint: Set literals appearing in one-element clauses to *true* with probability a (for some $1/2 < a < 1$) and assign the remaining variables to *true* with probability $1/2$. Derandomize this and choose a appropriately.

(4 points)

Exercise 3.2:
The KNAPSACK PROBLEM for an instance $\mathcal{I}$ can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \;\middle|\; \sum_{i=1}^{n} w_i x_i \le W,\, x_i \in \{0,1\}\, \forall 1 \le i \le n \right\} \tag{1}$$

Denote the optimum of (1) by $OPT(\mathcal{I})$ and let $LR(\mathcal{I})$ be the optimum of the linear relaxation where $x_i \in \{0,1\}$ is replaced by $0 \le x_i \le 1$.
Show that the *integrality gap* $\sup \left\{ \frac{LR(\mathcal{I})}{OPT(\mathcal{I})} \;\middle|\; OPT(\mathcal{I}) \neq 0 \right\}$ of the KNAPSACK PROBLEM is unbounded. Show that this doesn't remain valid if the definition of the KNAPSACK PROBLEM additionally requires $w_i \le W$ for all w_i.

(4 points)

Exercise 3.3:
Consider the FRACTIONAL MULTI KNAPSACK PROBLEM: Given natural numbers $n, m \in \mathbb{N}$ and w_i, c_{ij} as well as W_j for $1 \le i \le n$ and $1 \le j \le m$, find x_{ij} satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \le i \le n$ and $\sum_{i=1}^{n} x_{ij} w_i \le W_j$ for all $1 \le j \le m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum.
State a polynomial-time combinatorial algorithm for this problem or prove NP-hardness.

(4 points)

Exercise 3.4:
Describe exact algorithms with running time $O\left(2^{n/2}\right)$ for the following problems:

1. Subset Sum: Given $K, n, x_1, \ldots, x_n \in \mathbb{N}$, find $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = K$.
2. Knapsack Problem (where n denotes the number of items).

(4 points)

Please return your solutions before the lecture at Tuesday, **April 30th, 2:15 PM**.