Research Institute for Discrete Mathematics
Approximation Algorithms
Summer term 2012

Prof. Dr. S. Hougardy
Dipl.-Math. U. Suhl
D. Rotter

Exercise Set 9

STEINER POINT MINIMIZATION PROBLEM (SPMP)

Instance A finite set P with an embedding $\xi : P \to \mathbb{R}^2$, a "length bound" $\gamma \in \mathbb{R}_{>0}$, and a "Steiner bound" $s \in \mathbb{N}$.

Question Is there a set $Q \supseteq P$ with an embedding $\xi' : Q \to \mathbb{R}^2$ (satisfying $\xi'(v) = \xi(v)$ for all $v \in P$) and a Steiner tree T for P in the complete graph on Q such that $d(e) \leq \gamma$ for each edge $e \in E(T)$ and $|Q \setminus P| \leq s$? The length $d(\{v, w\})$ of an edge $\{v, w\}$ is defined as the Euclidean distance between $\xi'(v)$ and $\xi'(w)$. The elements of $Q \setminus P$ are called *Steiner points* and the elements of P *terminals*.

The SPMP is NP-complete. We consider the optimization version of the problem, i.e. finding a feasible solution with the least possible number of Steiner points. The goal of this exercise is to find a 5-approximation of the problem. For this, consider the following algorithm:

Algorithm 1 APPROXIMATESPMP

1: $T \leftarrow$ Minimum spanning tree over P
2: Insert $\lceil \frac{d(e)}{\gamma} \rceil - 1$ degree-2 Steiner points into each edge $e \in E(T)$ to subdivide it into equal pieces
3: Return the resulting tree T_A

T_A is clearly a Steiner tree on the terminal set P. Several steps are required to analyze the algorithm. We introduce the following notions: For a Steiner tree T, let $\#(T)$ be the number of Steiner points in T. An optimal solution T of the optimization version of SPMP that minimizes $d(T) := \sum_{e \in E(T)} d(e)$ among all optimal solutions is called *shortest optimal solution*.

(please turn over)

Exercise 1:
Prove the following lemmas:

(i) If T is a Steiner tree on the terminals P without Steiner points of degree more than 2, then $\#(T_A) \leq \#(T)$.

(ii) There is a shortest optimal solution T^* of SPMP such that all Steiner points in T^* have degree at most 5. (First, exclude the possibility of Steiner points with degree more than 6. Then analyze a degree-6 Steiner point and one of its neighbours.)

(2+4 points)

Exercise 2:
Use exercise 1 to show that ApproximateSPMP is a 5-approximation algorithm of SPMP. (It is sometimes useful to split a Steiner tree into its "full components", i.e. maximal subtrees such that all terminals are leaves.)

(4 points)

A more careful analysis of the algorithm shows that it is even a 4-approximation algorithm, but not better:

Exercise 3:
Define a graph with $\#(T_A) = 4 \cdot \#(T^*)$, where T^* is an optimal solution.

(4 points)

Please return the exercises until Tuesday, **June 12th, at 2:15 pm**.

The Research Institute for Discrete Mathematics is looking for **student assistants** to mentor exercise classes during the winter term 2012/2013. More information on

http://www.or.uni-bonn.de/news/tutoren_ws2012.html