Exercise Set 9

**Steiner Point Minimization Problem (SPMP)**

**Instance** A finite set $P$ with an embedding $\xi : P \to \mathbb{R}^2$, a “length bound” $\gamma \in \mathbb{R}_{>0}$, and a “Steiner bound” $s \in \mathbb{N}$.

**Question** Is there a set $Q \supseteq P$ with an embedding $\xi' : Q \to \mathbb{R}^2$ (satisfying $\xi'(v) = \xi(v)$ for all $v \in P$) and a Steiner tree $T$ for $P$ in the complete graph on $Q$ such that $d(e) \leq \gamma$ for each edge $e \in E(T)$ and $|Q \setminus P| \leq s$? The length $d\{v, w\}$ of an edge $\{v, w\}$ is defined as the Euclidean distance between $\xi'(v)$ and $\xi'(w)$. The elements of $Q \setminus P$ are called Steiner points and the elements of $P$ terminals.

The SPMP is $NP$-complete. We consider the optimization version of the problem, i.e. finding a feasible solution with the least possible number of Steiner points. The goal of this exercise is to find a 5-approximation of the problem. For this, consider the following algorithm:

**Algorithm 1 ApproximateSPMP**

1: $T \leftarrow$ Minimum spanning tree over $P$
2: Insert $\lceil \frac{d(e)}{\gamma} \rceil - 1$ degree-2 Steiner points into each edge $e \in E(T)$ to subdivide it into equal pieces
3: Return the resulting tree $T_A$

$T_A$ is clearly a Steiner tree on the terminal set $P$. Several steps are required to analyze the algorithm. We introduce the following notions: For a Steiner tree $T$, let $\#(T)$ be the number of Steiner points in $T$. An optimal solution $T$ of the optimization version of SPMP that minimizes $d(T) := \sum_{e \in E(T)} d(e)$ among all optimal solutions is called shortest optimal solution.

(please turn over)
Exercise 1:
Prove the following lemmas:

(i) If $T$ is a Steiner tree on the terminals $P$ without Steiner points of degree more than 2, then $(T_A) \leq #(T)$.

(ii) There is a shortest optimal solution $T^*$ of SPMP such that all Steiner points in $T^*$ have degree at most 5. (First, exclude the possibility of Steiner points with degree more than 6. Then analyze a degree-6 Steiner point and one of its neighbours.)

(2+4 points)

Exercise 2:
Use exercise 1 to show that APPROXIMATESPMP is a 5-approximation algorithm of SPMP. (It is sometimes useful to split a Steiner tree into its “full components”, i.e. maximal subtrees such that all terminals are leaves.)

A more careful analysis of the algorithm shows that it is even a 4-approximation algorithm, but not better:

Exercise 3:
Define a graph with $(T_A) = 4 \cdot #(T^*)$, where $T^*$ is an optimal solution.

(4 points)

Please return the exercises until Tuesday, June 12th, at 2:15 pm.

The Research Institute for Discrete Mathematics is looking for student assistants to mentor exercise classes during the winter term 2012/2013. More information on

http://www.or.uni-bonn.de/news/tutoren_ws2012.html