Scheduling
Parallel Machine Scheduling

Tim Nieberg

Parallel machine models: Makespan Minimization

Problem $P||C_{max}$:

- m machines
- n jobs with processing times $p_1, \ldots, p_n$

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Problem $P||C_{max}$:

- m machines
- n jobs with processing times $p_1, \ldots, p_n$
- variable $x_{ij} = \begin{cases} 1 & \text{if job } j \text{ is processed on machine } i \\ 0 & \text{else} \end{cases}$
- ILP formulation:

$$
\begin{array}{llcll}
\min & C_{max} & & & \\
s.t. & \sum_{j=1}^{n} x_{ij} p_j & \leq & C_{max} & i = 1, \ldots, m \\
 & \sum_{i=1}^{m} x_{ij} & = & 1 & j = 1, \ldots, n \\
 & x_{ij} & \in & \{0, 1\} & i = 1, \ldots, m; j = 1, \ldots, n
\end{array}
$$

Parallel machine models: Makespan Minimization

<u>Problem $P||C_{max}$</u>:

- in lecture 2: $P2||C_{max}$ is NP-hard
- $P||C_{max}$ is even NP-hard in the strong sense (reduction from 3-PARTITION); i.e. also pseudopolynomial algorithms are unlikely
- question: What happens if $x_{ij} \in \{0, 1\}$ in the ILP is relaxed?

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  answer: objective value of LP gets $\sum_{j=1}^{n} p_j/m$
- question: is this the optimal value of $P|pmtn|C_{max}$?

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- question: is this the optimal value of $P|pmtn|C_{max}$?
  answer: No!
  Example: $m = 2, n = 2, p = (1, 2)$

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  answer: objective value of LP gets $\sum_{j=1}^{n} p_j/m$
- question: is this the optimal value of $P|pmtn|C_{max}$?
  answer: No!
  Example: $m = 2, n = 2, p = (1,2)$
- add $C_{max} \geq p_j$ for $j = 1, \ldots, m$ to ensure that each job has enough time

LP for problem $P|pmtn|C_{max}$:

$$\begin{array}{llcll}
\min & C_{max} & & & \\
s.t. & \sum_{j=1}^{n} x_{ij} p_j & \leq & C_{max} & i = 1, \ldots, m \\
 & p_j & \leq & C_{max} & j = 1, \ldots, n \\
 & \sum_{i=1}^{m} x_{ij} & = & 1 & j = 1, \ldots, n \\
 & x_{ij} & \geq & 0 & i = 1, \ldots, m; j = 1, \ldots, n
\end{array}$$

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 & x_{ij} & \geq & 0 & i = 1, \ldots, m; j = 1, \ldots, n
\end{array}$$

- Optimal value of LP is $\max\{\max_{j=1}^{n} p_j, \sum_{j=1}^{n} p_j/m\}$
- LP gives no schedule, thus only a lower bound!
- construction of schedule: simple (page -4-) or via open shop (later)

Parallel machine models: Makespan Minimization

Wrap around rule for problem $P|pmtn|C_{max}$:

- define $opt := \max\{\max_{j=1}^{n} p_j, \sum_{j=1}^{n} p_j/m\}$
- opt is a lower bound on the optimal value for problem $P|pmtn|C_{max}$
- Construction of a schedule with $C_{max} = opt$:
  fill the machines successively, schedule the jobs in any order and preempt a job if the time bound opt is met
- all jobs can be scheduled since $opt \geq \sum_{j=1}^{n} p_j/m$
- no job is scheduled at the same time on two machines since $opt \geq \max_{j=1}^{n} p_j$

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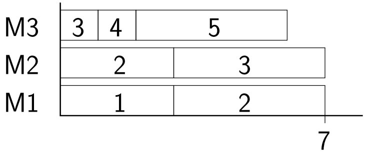
- no job is scheduled at the same time on two machines since $opt \geq \max_{j=1}^{n} p_j$
- Example: $m = 3, n = 5, p = (3, 7, 5, 1, 4)$



Schedule construction via Open shop for $P|pmtn|C_{max}$:

- given an optimal solution x of the LP, consider the following open shop instance
  - n jobs, m machines and $p_{ij} := x_{ij} p_j$
- solve for this instance $O|pmtn|C_{max}$

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<u>Schedule construction via Open shop for $P|pmtn|C_{max}$</u>:

- given an optimal solution x of the LP, consider the open shop instance n jobs, m machines and $p_{ij} := x_{ij} p_j$
- solve for this instance $O|pmtn|C_{max}$
- <u>Result</u>: solution for problem $P|pmtn|C_{max}$
- for $O|pmtn|C_{max}$ we show later that an optimal solution has value
$$\max\{\max_{j=1}^{n} \sum_{i=1}^{m} p_{ij}, \max_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\}$$
and can be calculated in polynomial time
- <u>Result</u>: solution of $O|pmtn|C_{max}$ is optimal for $P|pmtn|C_{max}$

Uniform machines: $Q|pmtn|C_{max}$:

- m machines with speeds $s_1, \ldots, s_m$
- n jobs with processing times $p_1, \ldots, p_n$
- change LP!

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Uniform machines: $Q|pmtn|C_{max}$:

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- n jobs with processing times $p_1, \ldots, p_n$

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\min & C_{max} & & & \\
s.t. & \sum_{j=1}^{n} x_{ij} p_j / s_i & \leq & C_{max} & i = 1, \ldots, m \\
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\end{array}
$$

Uniform machines: $Q|pmtn|C_{max}$ (cont.):

- since again no schedule is given, LP leads to lower bound for optimal value of $Q|pmtn|C_{max}$,
- as for $P|pmtn|C_{max}$ we may solve an open shop instance corresponding to the optimal solution x of the LP with n jobs, m machines and $p_{ij} := x_{ij} p_j / s_i$
- this solution is an optimal schedule for $Q|pmtn|C_{max}$

Parallel machine models: Makespan Minimization

<u>Unrelated machines: $R|pmtn|C_{max}$</u>:

- m machines
- n jobs with processing times $p_1, \ldots, p_n$
- speed s_{ij}
- change LP!

Parallel machine models: Makespan Minimization

Unrelated machines: $R|pmtn|C_{max}$:

- m machines
- n jobs with processing times $p_1, \ldots, p_n$ and given speeds s_{ij}

$$
\begin{array}{lrcll}
\min & C_{max} & & & \\
s.t. & \sum_{j=1}^{n} x_{ij} p_j / s_{ij} & \leq & C_{max} & i = 1, \ldots, m \\
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\end{array}
$$

Unrelated machines: $R|pmtn|C_{max}$ (cont.):

- same procedure as for $Q|pmtn|C_{max}$!
  - again no schedule is given,
  - LP leads to lower bound for optimal value of $R|pmtn|C_{max}$,
  - for optimal solution x solve an corresponding open shop instance with n jobs, m machines and $p_{ij} := x_{ij}p_j/s_{ij}$
  - this solution is an optimal schedule for $R|pmtn|C_{max}$

Parallel machine models: Makespan Minimization

Approximation methods for: $P||C_{max}$:

- list scheduling methods (based on priority rules)
  - jobs are ordered in some sequence π
  - always when a machine gets free, the next unscheduled job in π is assigned to that machine
- Theorem: List scheduling is a $(2 - 1/m)$-approximation for problem $P||C_{max}$ for any given sequence π
- Proof on the board
- Holds also for $P|r_j|C_{max}$

Parallel machine models: Makespan Minimization

Approximation methods for: $P||C_{max}$ (cont.):

- consider special list
- LPT-rule (longest processing time first) is a natural candidate
- Theorem: The LPT-rule leads to a $(4/3 - 1/3m)$-approximation for problem $P||C_{max}$
  - Proof on the board uses following result:
  - Lemma: If an optimal schedule for problem $P||C_{max}$ results in at most 2 jobs on any machine, then the LPT-rule is optimal
  - Proof as Exercise
- the bound $(4/3 - 1/3m)$ is tight (Exercise)

Parallel machine models: Total Completion Time

Parallel machines: $P||\sum C_j$:

- for $m = 1$, the SPT-rule is optimal (see Lecture 2)
- for $m \geq 2$ a partition of the jobs is needed
- if a job j is scheduled as k-last job on a machine, this job contributes kp_j to the objective value

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- if a job j is scheduled as k-last job on a machine, this job contributes kp_j to the objective value
- we have m last positions where the processing time is weighted by 1, m second last positions where the processing time is weighted by 2, etc.
- use the n smallest weights for positioning the jobs

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- if a job j is scheduled as k-last job on a machine, this job contributes kp_j to the objective value
- we have m last positions where the processing time is weighted by 1, m second last positions where the processing time is weighted by 2, etc.
- use the n smallest weights for positioning the jobs
- assign job with the ith largest processing time to ith smallest weight is optimal
- Result: SPT is also optimal for $P||\sum C_j$

Uniform machines: $Q||\sum C_j$:

- if a job j is scheduled as k-last job on a machine M_r, this job contributes $kp_j/s_r = (k/s_r)p_j$ to the objective value; i.e. job j gets 'weight' (k/s_r)
- for scheduling the n jobs on the m machines, we have weights

$$\{\frac{1}{s_1}, \ldots, \frac{1}{s_m}, \frac{2}{s_1}, \ldots, \frac{2}{s_m}, \ldots, \frac{n}{s_1}, \ldots, \frac{n}{s_m}\}$$

- from these nm weights we select the n smallest weights and assign the ith largest job to the ith smallest weight leading to an optimal schedule

Parallel machine models: Total Completion Time

Example uniform machines: $Q||\sum C_j$:

- $n = 6,\ \ p = (6, 9, 8, 12, 4, 2)$
- $m = 3,\ \ s = (3, 1, 4)$
- possible weights:

$$\{\frac{1}{3}, \frac{1}{1}, \frac{1}{4}, \frac{2}{3}, \frac{2}{1}, \frac{2}{4}, \frac{3}{3}, \frac{3}{1}, \frac{3}{4}, \frac{4}{3}, \frac{4}{1}, \frac{4}{4}, \frac{5}{3}, \frac{5}{1}, \frac{5}{4}, \frac{6}{3}, \frac{6}{1}, \frac{6}{4}\}$$

- 6 smallest weights:

$$\{\color{red}{\frac{1}{3}}, \frac{1}{1}, \color{red}{\frac{1}{4}}, \color{red}{\frac{2}{3}}, \frac{2}{1}, \color{red}{\frac{2}{4}}, \frac{3}{3}, \frac{3}{1}, \color{red}{\frac{3}{4}}, \frac{4}{3}, \frac{4}{1}, \color{red}{\frac{4}{4}}, \frac{5}{3}, \frac{5}{1}, \frac{5}{4}, \frac{6}{3}, \frac{6}{1}, \frac{6}{4}\}$$

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- $n = 6, \; p = (6, 9, 8, 12, 4, 2)$
- $m = 3, \; s = (3, 1, 4)$
- 6 smallest weights:

$$\{\frac{1}{3}, \frac{1}{1}, \frac{1}{4}, \frac{2}{3}, \frac{2}{1}, \frac{2}{4}, \frac{3}{3}, \frac{3}{1}, \frac{3}{4}, \frac{4}{3}, \frac{4}{1}, \frac{4}{4}, \frac{5}{3}, \frac{5}{1}, \frac{5}{4}, \frac{6}{3}, \frac{6}{1}, \frac{6}{4}\}$$

- sorted list of weights:

$$\{\frac{1}{4}, \frac{1}{3}, \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4}\}$$

- jobs sorted by decreasing processing times: $(4, 2, 3, 1, 5, 6)$

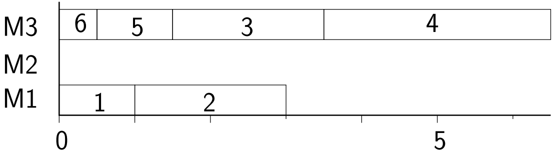
Parallel machine models: Total Completion Time

Example uniform machines: $Q||\sum C_j$:

- $n = 6, \ p = (6, 9, 8, 12, 4, 2)$
- $m = 3, \ s = (3, 1, 4)$
- sorted list of weights:

$$\{\frac{1}{4}, \frac{1}{3}, \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4}\}$$

- jobs sorted by decreasing processing times: $(4, 2, 3, 1, 5, 6)$
- Schedule:



Unrelated machines: $R||\sum C_j$:

- if a job j is scheduled as k-last job on a machine M_r, this job contributes kp_j/s_{rj} to the objective value;
- since now the 'weight' is also job-dependent, we cannot simply sort the 'weights'
- assignment problem:
  - n jobs
  - nm machine positions (k, r) (k-last position on M_r)
  - assigning job j to (k, r) has costs kp_j/s_{rj}
  - find an assignment of minimal costs of all jobs to machine positions
- leads to optimal solution of $R||\sum C_j$ in polynomial time

Parallel machine models: Total Weighted Completion Time

Parallel machines: $P||\sum w_j C_j$:

- Problem $1||\sum w_j C_j$ is solvable via the WSPT-rule (Lecture 2)
- Problem $P2||\sum w_j C_j$ is ...

Parallel machine models: Total Weighted Completion Time

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- Problem $1||\sum w_jC_j$ is solvable via the WSPT-rule (Lecture 2)
- Problem $P2||\sum w_jC_j$ is already NP-hard, but
- Problem $P2||\sum w_jC_j$ is pseudopolynomial solvable
- Problem $P||\sum w_jC_j$ is NP-hard in the strong sense
  Proof by reduction using 3-PARTITION as exercise
- Approximation:

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- Problem $P2||\sum w_j C_j$ is pseudopolynomial solvable
- Problem $P||\sum w_j C_j$ is NP-hard in the strong sense
  Proof by reduction using 3-PARTITION as exercise
- Approximation: the WSPT-rule gives an $\frac{1}{2}(1+\sqrt{2})$ approximation
  Proof is not given; uses fact that worst case examples have equal w_j/p_j ratios for all jobs