1) Given an instance of the Min-Max Resource Sharing Problem, $\sigma$-approximate bounded block solvers are oracles $f_c : \mathbb{R}_+^{E_c} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $y \in \mathbb{R}_+^R$ and $\mu \leq 0$ we have $g_c(f_c(y, \mu)) \leq \mu \mathbf{1}$ and $y^T g_c(f_c(y, \mu)) \leq \sigma \inf \{y^T g_c(b) \mid b \in B_c, g_c(b) \leq \mu \mathbf{1}\}$ for all $c \in \mathcal{C}$ (where $\mathbf{1}$ is the all-one vector). For $\mu = \infty$ we have the usual unbounded block solvers.

Show that the Resource Sharing Algorithm can be modified for instances with $\lambda^* \leq 1$ and bounded block solvers such that it computes for $0 < \delta, \delta' < 1$ a $(\sigma (1 + \delta) + \delta')$-approximate solution in $O(\theta |\mathcal{C}| \log |\mathcal{R}| (\log |\mathcal{R}| + (\delta \delta')^{-1} \sigma))$ time (compare to Theorem 4.12). To this end show that at most $t |\mathcal{C}|$ oracle calls are sufficient.

(4 points)

2) Given a pair $(G, H)$ of undirected graphs on the same set of vertices, capacities $u : E(G) \rightarrow \mathbb{R}_+$ and demands $b : E(H) \rightarrow \mathbb{R}_+$.

A Concurrent Flow of value $\alpha > 0$ is a family $(x^f)_{f \in E(H)}$, where $x^f$ is an $s$-$t$-flow of value $\alpha \cdot b(f)$ in $(V(G), \{(v, w), (w, v) \mid \{v, w\} \in E(G)\})$ for each $f = \{t, s\} \in E(H)$, and

$$\sum_{f \in E(H)} (x^f((v, w)) + x^f((w, v))) \leq u(e)$$

for all $e = \{v, w\} \in E(G)$.

The Maximum Concurrent Flow Problem is to find a concurrent flow with maximum value $\alpha$.

Prove that the Maximum Concurrent Flow Problem is a special case of the Min-Max Resource Sharing Problem. How can you implement

a) unbounded block solvers and

b) bounded block solvers?

(4 points)
3) Let $k \geq 2$ and consider the following instance of the Resource Sharing Problem:

$C := \{c\}$, $\mathcal{R} := \{r_1, \ldots, r_{k+1}\}$, $\mathcal{B}_c := \text{conv}(\{b_1, b_2\})$ with

i) $g_c(b_1) := (1, 0, \ldots, 0)$,

ii) $g_c(b_2) := (0, 1, \ldots, 1)$,

iii) $g_c(\alpha b_1 + (1 - \alpha) b_2) := \alpha g_c(b_1) + (1 - \alpha) g_c(b_2)$ for $0 \leq \alpha \leq 1$.

Assume $\sigma = 1$, i.e. the block solvers $f_c, c \in C$, always returns an optimum solution. Show that computing a $(1 + \omega)$-approximate solution for this instance using the Resource Sharing Algorithm requires $\Theta(\omega^{-2 \ln k})$ phases.

(4 points)

4) Let $G$ be a directed graph with distances $l : E(G) \to \mathbb{R}_+$, $L \in V(G)$ a vertex that is reachable from any other vertex, and $t \in V(G)$. For $v \in V(G)$ let $d(v) := \max\{0, \text{dist}_l(v, L) - \text{dist}_l(v, t)\}$ (where $\text{dist}_l(v, w)$ denotes the length of a shortest $v$-$w$ path in $G$ with respect to $l$). For $e = (v, w) \in E(G)$ define $l'(e) := l(e) - d(v) + d(w)$. Let $s \in V(G)$ and $P$ a directed $s$-$t$ path in $G$. Prove that $l'(e) \geq 0$ for all $e \in E(G)$ and that $P$ is a shortest $s$-$t$ path with respect to $l$ if and only if it is a shortest $s$-$t$ path with respect to $l'$.

(4 points)

Deadline: July 13 before the lecture (12.15 pm).