Research Institute for Discrete Mathematics
Approximation Algorithms
Summer term 2010

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Exercise Set 6

Steiner Point Minimization Problem (SPMP)

Instance A finite set P with an embedding $\xi : P \to \mathbb{R}^2$, a "length bound" $\gamma \in \mathbb{R}_{>0}$, and a "Steiner bound" $s \in \mathbb{N}$.

Question Is there a set $Q \supseteq P$ with an embedding $\xi' : Q \to \mathbb{R}^2$ (satisfying $\xi'(v) = \xi(v)$ for all $v \in P$) and a Steiner tree T for P in the complete graph on Q such that $d(e) \leq \gamma$ for each edge $e \in E(T)$ and $|Q \setminus P| \leq s$? The length $d(\{v, w\})$ of an edge $\{v, w\}$ is defined as the Euclidean distance between $\xi'(v)$ and $\xi'(w)$. The elements of $Q \setminus P$ are called *Steiner points* and the elements of P *terminals*.

Exercise 1:
Show that SPMP is NP-complete. To do so, consider the following problem which you can assume to be NP-complete: Given a finite set X with an embedding $\psi : X \to \mathbb{Z}^2$ and a positive integer $L \in \mathbb{Z}_{>0}$, is there a set $Y \supseteq X$ with an embedding $\psi' : Y \to \mathbb{Z}^2$ (satisfying $\psi'(v) = \psi(v)$ for all $v \in X$) and a Steiner tree T for X in the complete graph on Y with $\sum_{e \in E(T)} \lceil d(e) \rceil \leq L$ (where $d(e)$ is defined as above)? Then describe a polynomial-time reduction from this problem to SPMP. (Use $\gamma := 1$ and $s := L - (|X| - 1)$.)

(4 points)

Now we consider the optimization version of the problem, i.e. finding a feasible solution with the least possible number of Steiner points. The goal of this exercise is to find a 5-approximation of the problem. For this, consider the following algorithm:

Algorithm 1 ApproximateSPMP

1: $T \leftarrow$ Minimum spanning tree over P
2: Insert $\lceil \frac{d(e)}{\gamma} \rceil - 1$ degree-2 Steiner points into each edge $e \in E(T)$ to subdivide it into equal pieces
3: Return the resulting tree T_A

(please turn over)

T_A is clearly a Steiner tree on the terminal set P. Several steps are required to analyze the algorithm. We introduce the following notions: For a Steiner tree T, let $\#(T)$ be the number of Steiner points in T. An optimal solution T of the optimization version of SPMP that minimizes $d(T) := \sum_{e \in E(T)} d(e)$ among all optimal solutions is called *shortest optimal solution*.

Exercise 2:
Prove the following lemmas:

(i) If T is a Steiner tree on the terminals P without Steiner points of degree more than 2, then $\#(T_A) \leq \#(T)$.

(ii) There is a shortest optimal solution T^* of SPMP such that all Steiner points in T^* have degree at most 5. (First, exclude the possibility of Steiner points with degree more than 6. Then, analyze a degree-6 Steiner point and one of its neighbours.)

(2+4 points)

Exercise 3:
Use exercise 2 to show that ApproximateSPMP is a 5-approximation algorithm of SPMP. (It is sometimes useful to split a Steiner tree into its "full components", i.e. maximal subtrees such that all terminals are leaves.)

(4 points)

A more careful analysis of the algorithm shows that it is even a 4-approximation algorithm, but not better:

Bonus Exercise 4:
Define a graph with $\#(T_A) = 4 \cdot \#(T^*)$, where T^* is an optimal solution.

(4 bonus points)

Special topic:
The next meeting of the institute's group of mentors takes place on Tuesday, June 1$^{\text{st}}$, at 6:00 pm in the conference room of the Arithmeum. The topic is "Introduction to Hypergraphs" and all interested students are invited. Hypergraphs are graphs where an edge can contain an arbitrary number of vertices.

Please return the exercises until Tuesday, **June 8th, at 2:15 pm**.