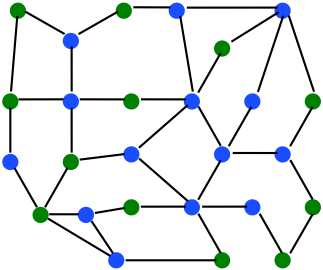
Faster Algorithm for Optimum Steiner Trees

Jens Vygen

University of Bonn

Aussois 2010

The Steiner Tree Problem



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- an undirected graph G
- with weights $c : E(G) \to \mathbb{R}_+$,
- and a terminal set $T \subseteq V(G)$,

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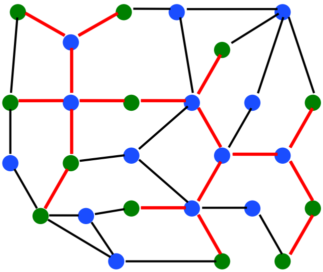

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- a Steiner tree Y for T in G
  (a tree which is a subgraph of G containing all terminals)
- such that $\sum_{e \in E(Y)} c(e)$ is minimum.

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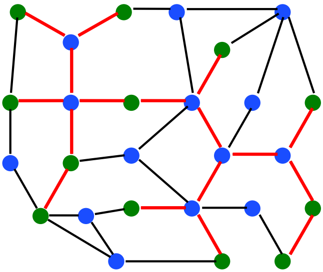

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- *NP*-hard (Karp [1972]) and even
- *MAXSNP*-hard (Bern, Plassmann [1989])

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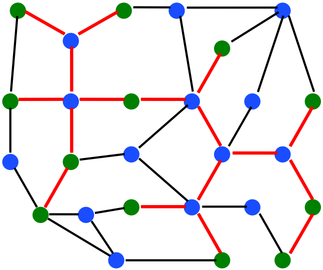

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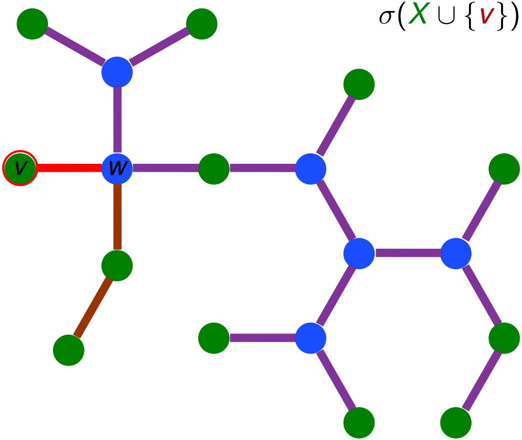
Sufficient to consider the metric closure.

The Dreyfus-Wagner Algorithm

For $\emptyset \neq X \subseteq T$ and $v \subseteq V(G) \setminus X$:

$$\sigma(X \cup \{v\}) = \sigma(X' \cup \{w\}) + \sigma((X \setminus X') \cup \{w\}) + c(\{v, w\})$$



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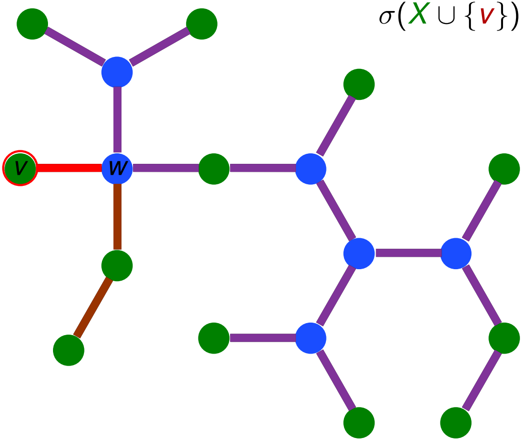

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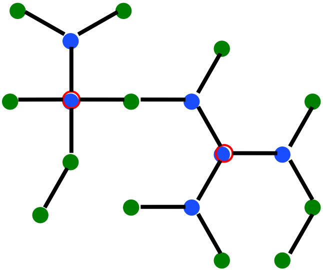
- due to Dreyfus and Wagner [1972] and Levin [1971]
- running time $O(3^k n + 2^k(m + n \log n))$, where $n = |V(G)|$, $m = |E(G)|$, $k = |T|$ (Erickson, Monma and Veinott [1987])

Recent Improvements

- Fuchs, Kern, Mölle, Richter, Rossmanith, Wang [2007]:
  Find a set X of vertices that split an optimum Steiner tree into subtrees with few terminals (by enumeration).
  $O\left((2+\delta)^k n^{(\ln(1/\delta)/\delta)^\zeta}\right)$ for $\zeta > \frac{1}{2}$ and sufficiently small $\delta > 0$.



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  This bound is at least
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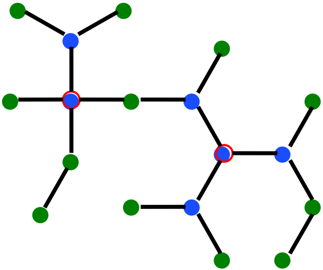

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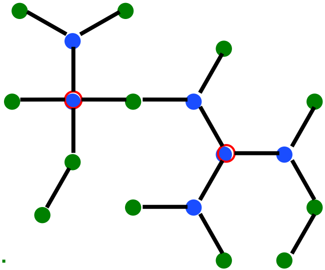

- Fomin, Grandoni, Kratsch [2008]:
  $O(5.96^k n^{O(\log k)})$-time algorithm with polynomial space
- Björklund, Husfeldt, Kaski, Koivisto [2007], Fomin, Grandoni, Kratsch [2008], Nederlof [2009]:
  algorithms for special case (edge weights are small integers)

Algorithms for Optimum Steiner Trees

- Erickson, Monma, Veinott [1987]: $O\left(3^k n + 2^k(m + n\log n)\right)$

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- enumeration: $O\left(m\alpha(m,n)2^{n-k}\right)$

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- here: $O\left(nk2^{k+(\log_2 k)(\log_2 n)}\right)$
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Algorithms for Optimum Steiner Trees

- Erickson, Monma, Veinott [1987]: $O\left(3^k n + 2^k(m + n\log n)\right)$
  fastest if $k < 4\log n$

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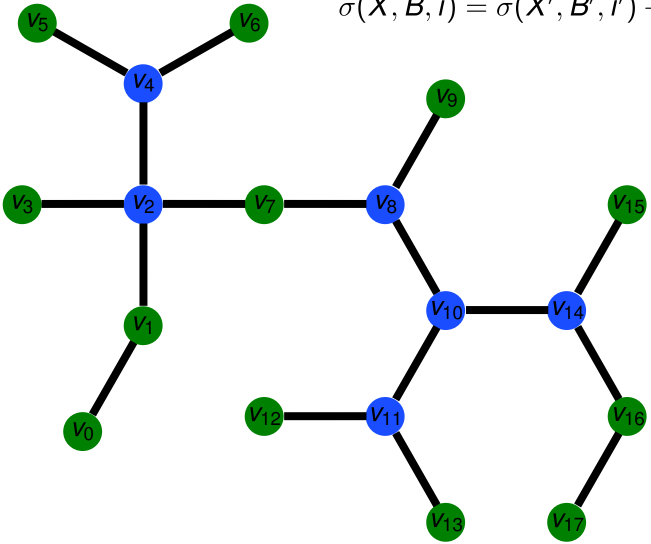
- enumeration: $O\left(m\alpha(m,n)2^{n-k}\right)$
  fastest if $k > (n - \log^2 n)/2$

New Algorithm

For $\emptyset \neq X \subseteq T$ and $B \subseteq V(G) \setminus T$:
$\sigma(X, B, i) = \sigma(X', B', i') + c(e)$.



$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}, v_{15}, v_{16}, v_{17}\}, \emptyset, 1)$

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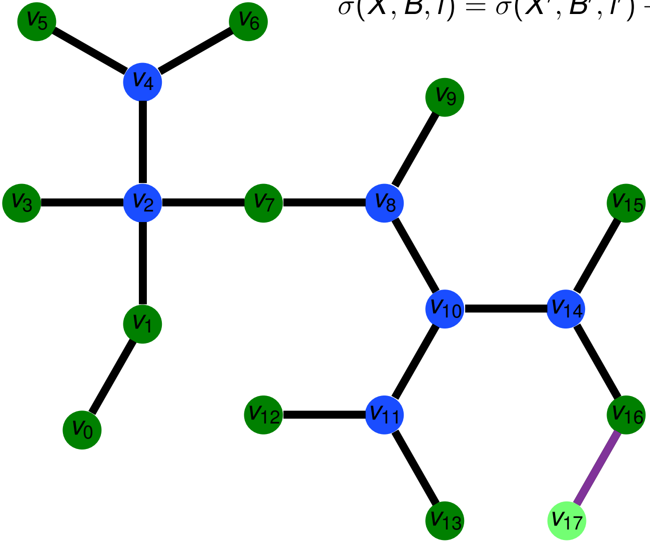

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}, v_{15}, v_{16}, v_{17}\}, \emptyset, 1) = c(\{v_{17}, v_{16}\}) +$
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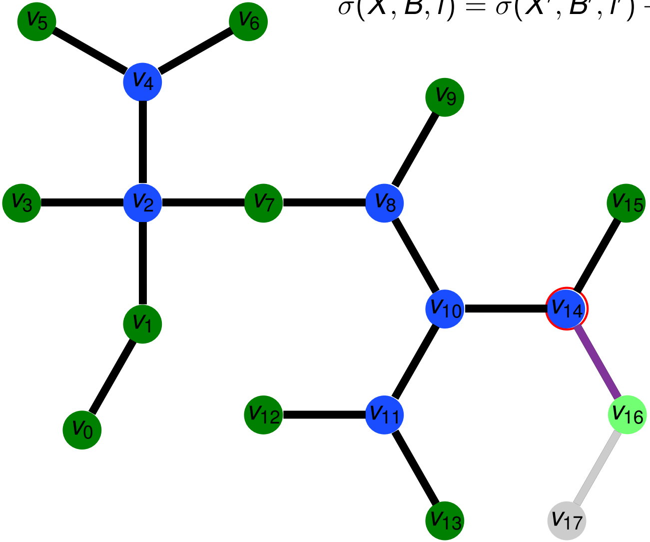

$$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}, v_{15}, v_{16}\}, \emptyset, 1) = c(\{v_{16}, v_{14}\}) +$$
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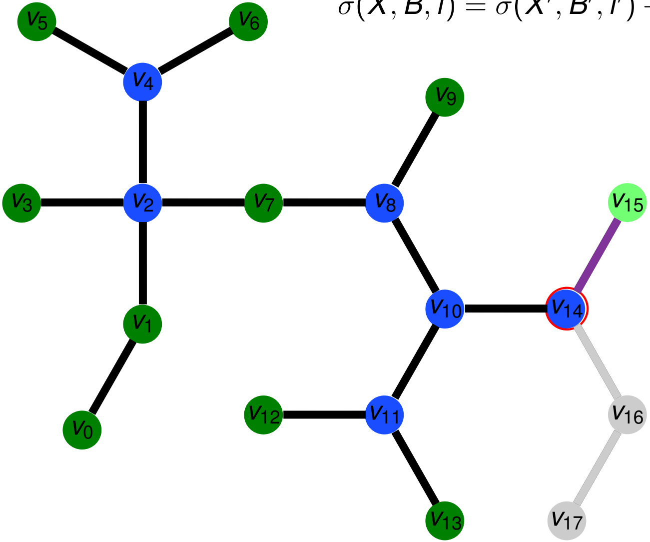

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}, v_{15}\}, \{v_{14}\}, 1) = c(\{v_{15}, v_{14}\}) +$
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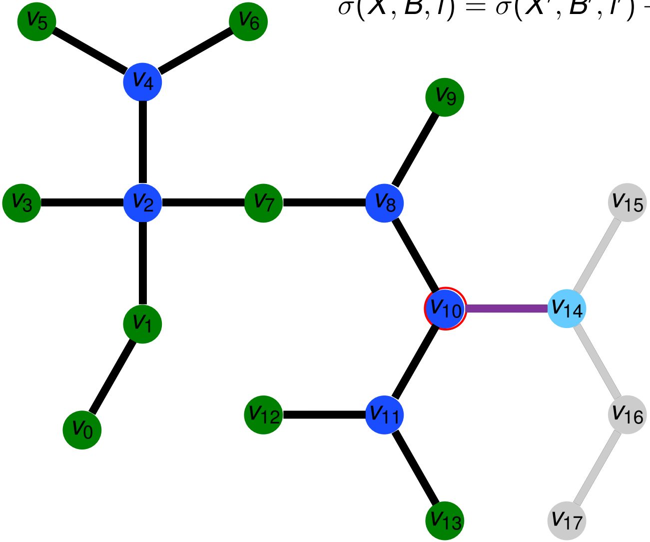

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}\}, \{v_{14}\}, 2) = c(\{v_{14}, v_{10}\}) +$
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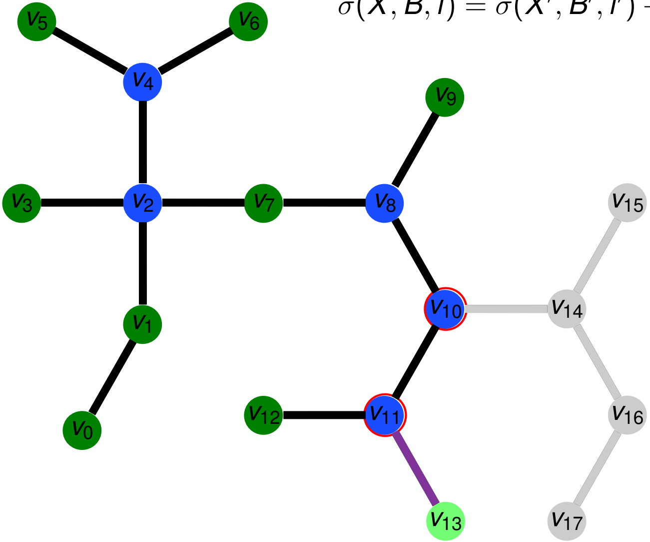

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9, v_{12}, v_{13}\}, \{v_{10}\}, 1) = c(\{v_{13}, v_{11}\}) +$
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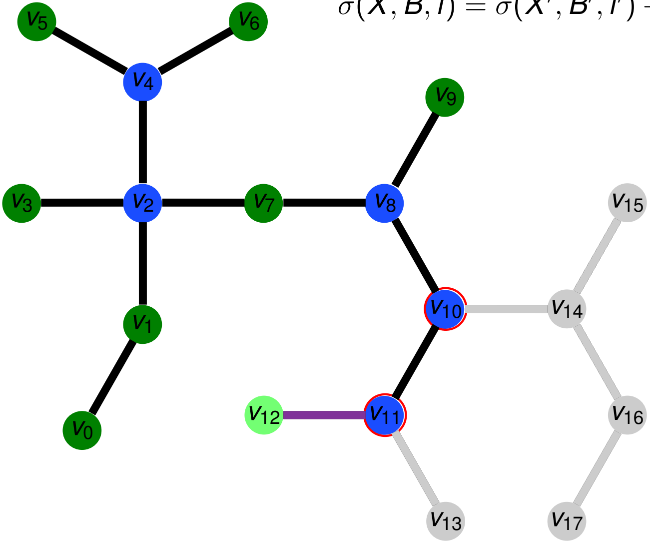

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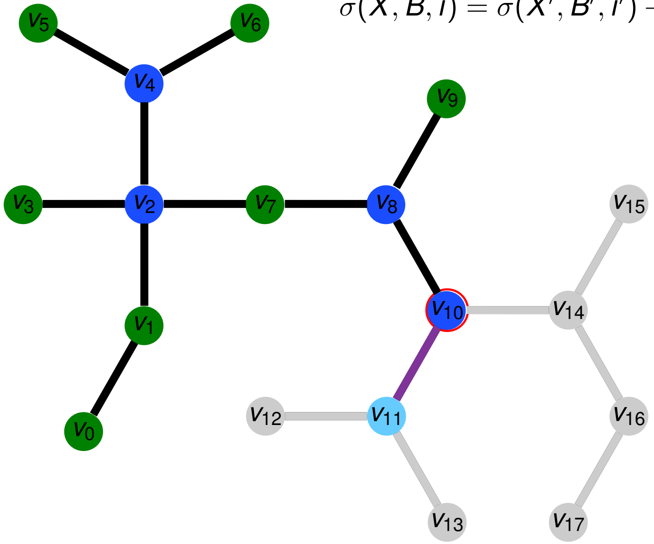

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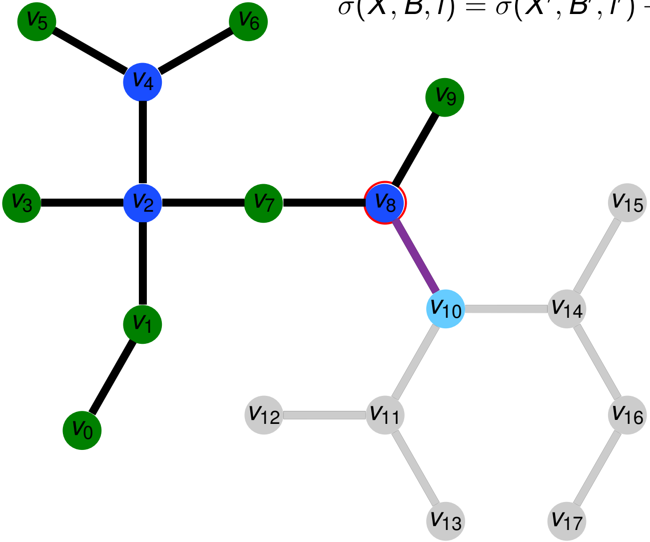

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7, v_9\}, \{v_{10}\}, 2) = c(\{v_{10}, v_8\}) +$
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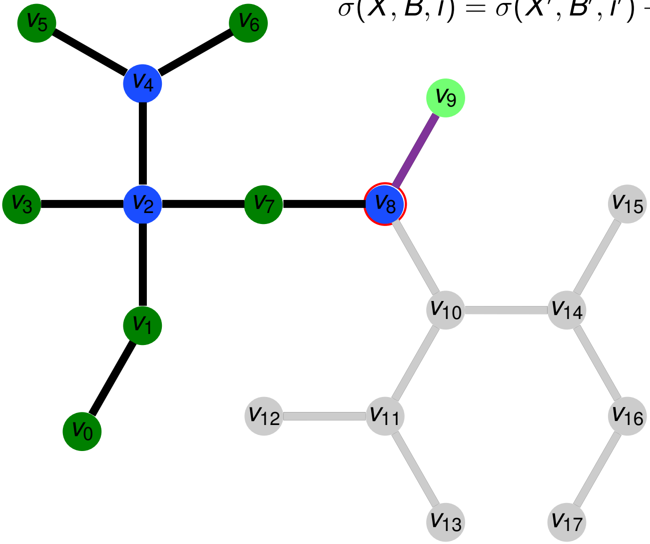

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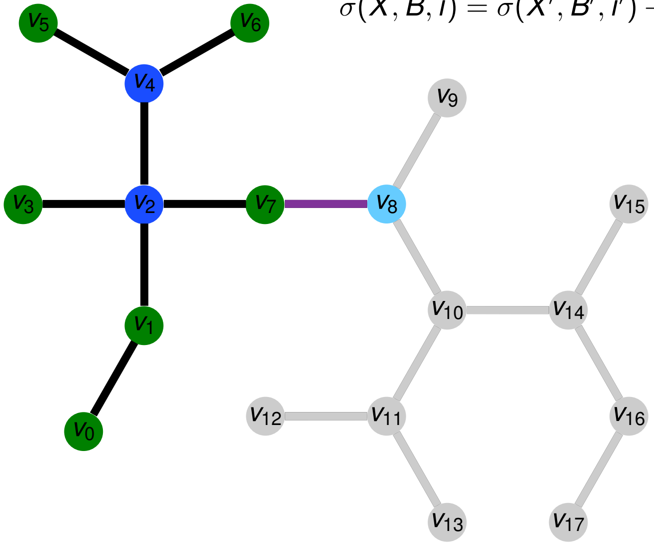

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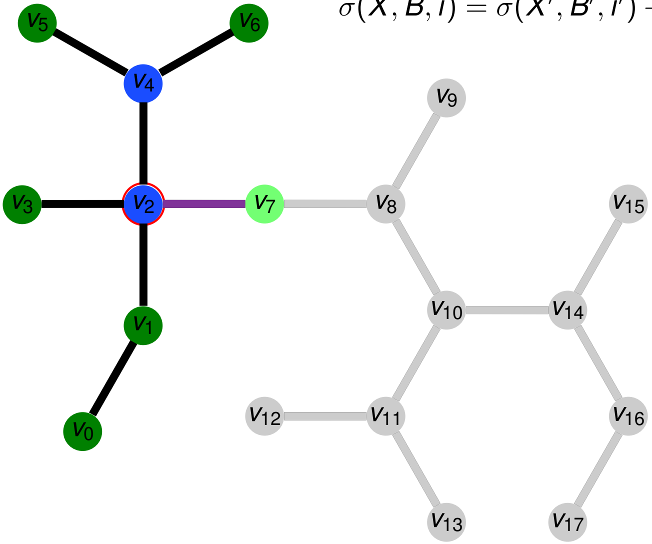

$\sigma(\{v_0, v_1, v_3, v_5, v_6, v_7\}, \emptyset, 1) = c(\{v_7, v_2\})+$
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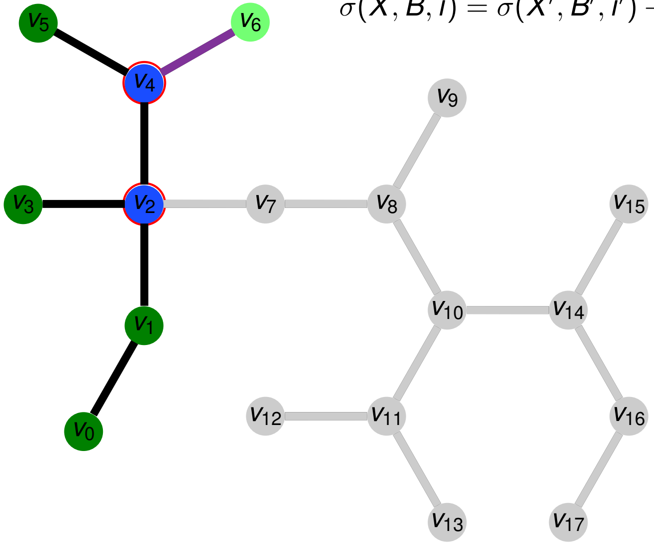

$\sigma(\{v_0, v_1, v_3, v_5, v_6\}, \{v_2\}, 1) = c(\{v_6, v_4\}) +$
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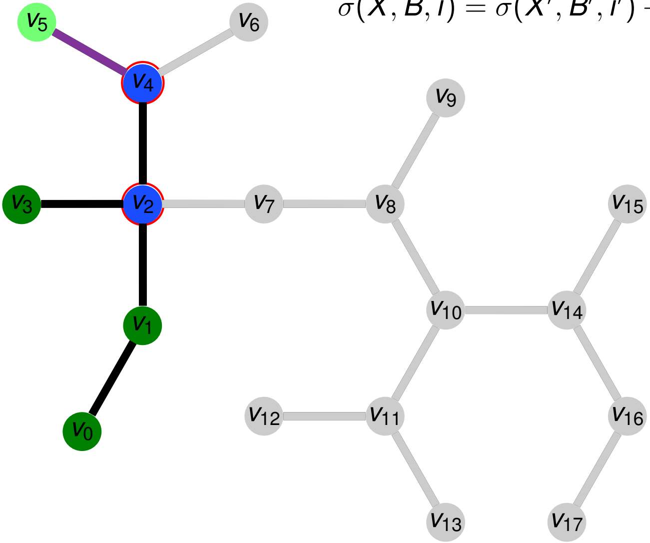

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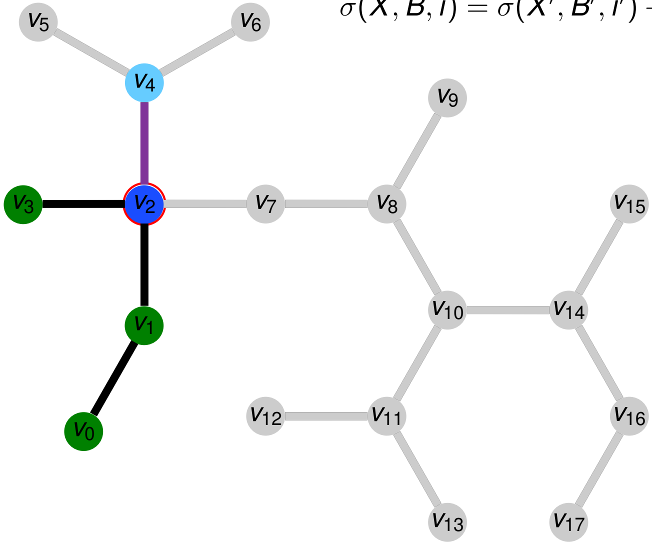

$\sigma(\{v_0, v_1, v_3\}, \{v_2, v_4\}, 2) = c(\{v_4, v_2\}) +$
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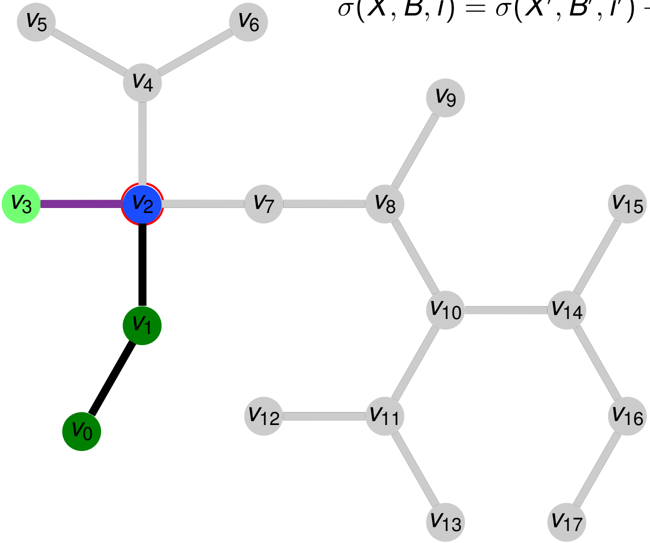

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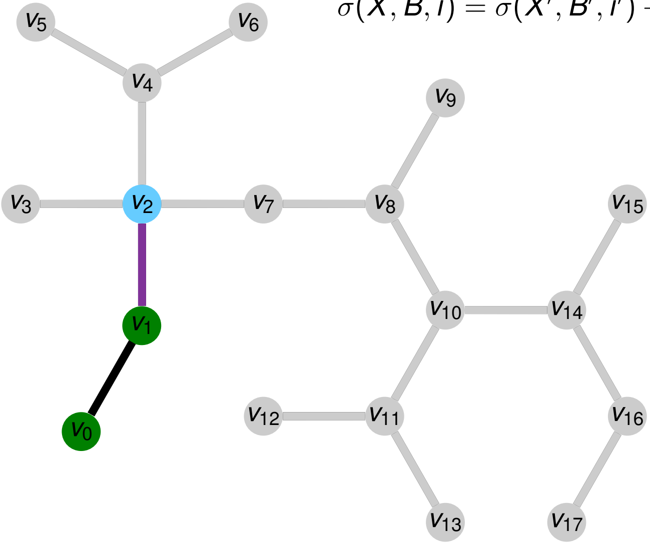

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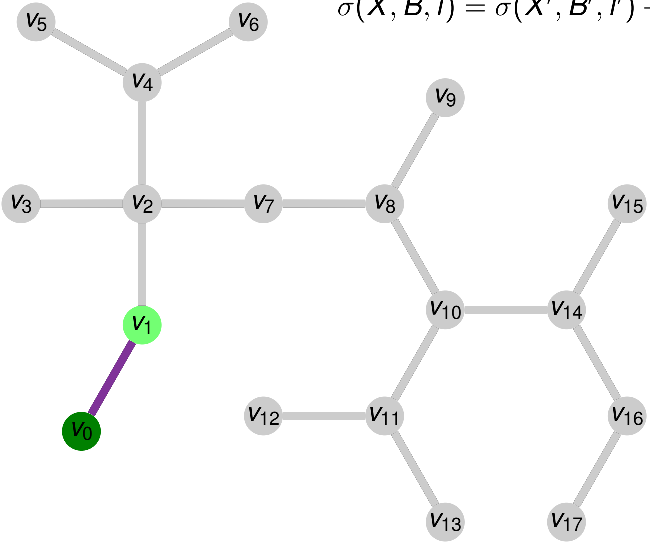

$\sigma(\{v_0, v_1\}, \emptyset, 1) = c(\{v_1, v_0\}) +$
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Recursion formulas

For $\emptyset \neq X \subseteq T$ and $B \subseteq V(G) \setminus T$:

$$\sigma(X, B, 1) := \begin{cases} 0 & B = \emptyset \\ \infty & B \neq \emptyset \end{cases} \qquad \text{if } |X| = 1,$$

$$\sigma(X, B, 1) := \min \Big\{ \min\{\sigma(X \setminus \{x\}, B, i) + c(u, x) : x \in X,\ u \in (X \setminus \{x\}) \cup B,\ i \in \{1, 2\}\},$$

$$\min\{\sigma(X \setminus \{x\}, B \cup \{u\}, 1) + c(u, x) : x \in X,\ u \in V(G) \setminus (X \cup B)\} \Big\} \qquad \text{if } |X| \geq 2,$$

$$\sigma(X, B, 2) := \infty \qquad \text{if } B = \emptyset,$$

$$\sigma(X, B, 2) := \min \Big\{ \min\{\sigma(X, B \setminus \{b\}, i) + c(u, b) : b \in B,\ u \in X \cup (B \setminus \{b\}),\ i \in \{1, 2\}\},$$

$$\min\{\sigma(X, B \setminus \{b\} \cup \{u\}, 1) + c(u, b) : b \in B,\ u \in V(G) \setminus (T \cup B)\} \Big\} \qquad \text{if } B \neq \emptyset.$$

Observation

Theorem

Let G be a complete graph. Let $c : E(G) \to \mathbb{R}_+$ satisfy the triangle inequality. Let $T \subseteq V(G)$.
Then $\sigma(T, \emptyset, 1)$ is the minimum length of a Steiner tree for T.

How many boundary points are needed?

Theorem

Let Y be a tree and $T \subseteq V(Y)$ such that each vertex in $V(Y) \setminus T$ has degree at least three. Let $n := |V(Y)|$, $k := |T| \geq 3$, and

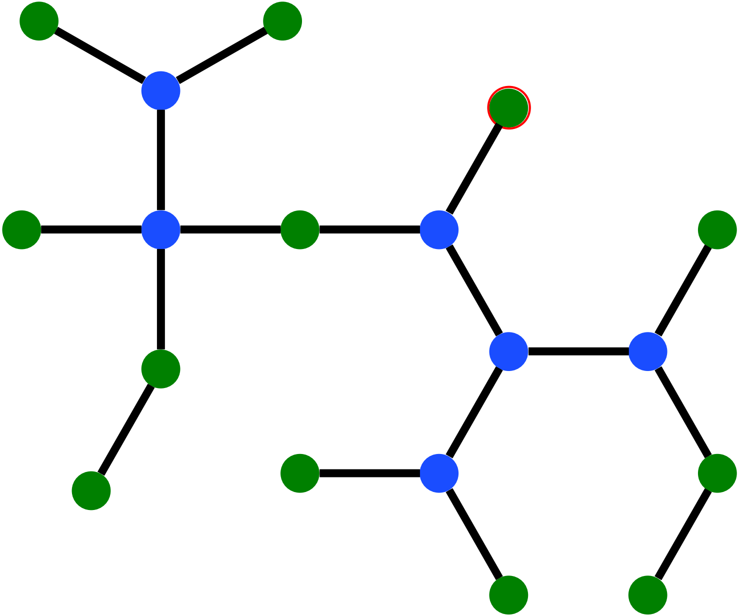
$$\beta := 1 + \lfloor \log_2(k/3) \rfloor.$$

Then we can number $V(Y) = \{v_0, \ldots, v_{n-1}\}$ and $E(Y) = \{e_1, \ldots, e_{n-1}\}$ such that the following conditions hold:

$$v_i \in e_i \subseteq \{v_0, \ldots, v_i\} \quad \text{for } i = 1, \ldots, n-1.$$

$$\left| \{v_0, \ldots, v_i\} \cap \bigcup_{h=i+1}^{n-1} e_h \right| \leq \beta \quad \text{for } i = 0, \ldots, n-1.$$

Proof (Sketch)



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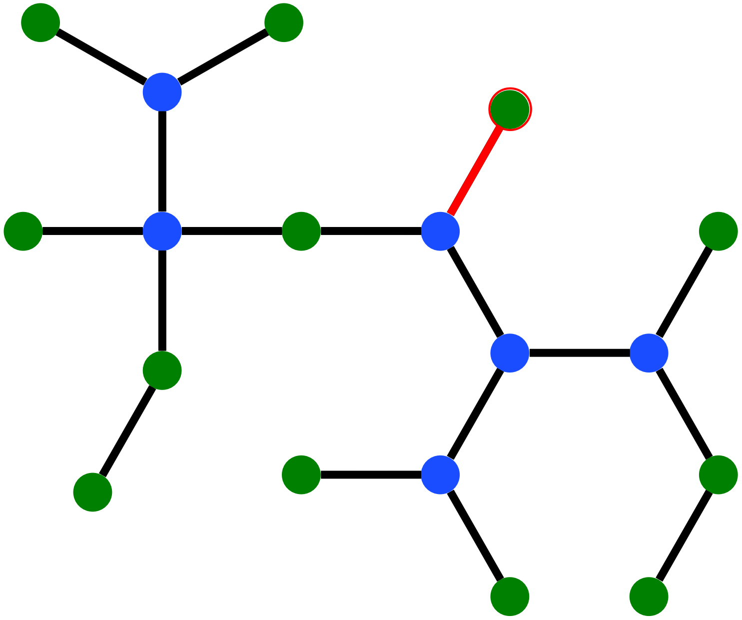
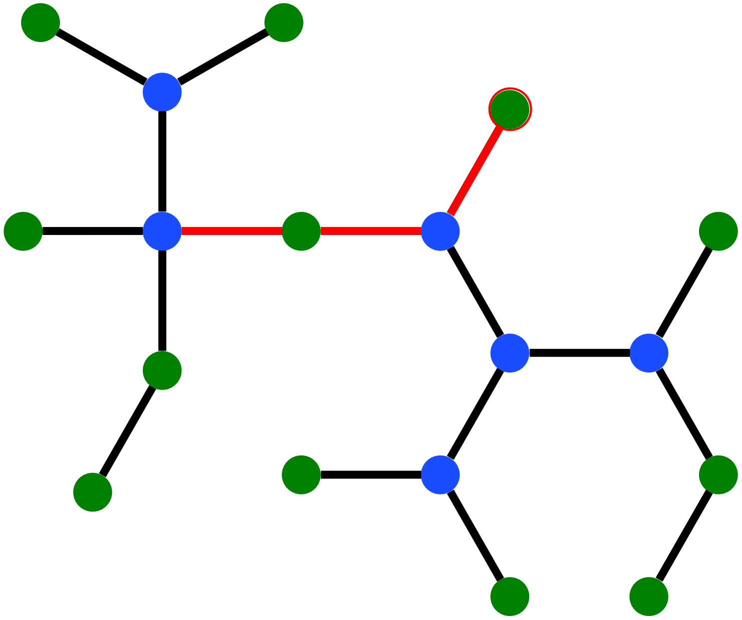


Proof (Sketch)



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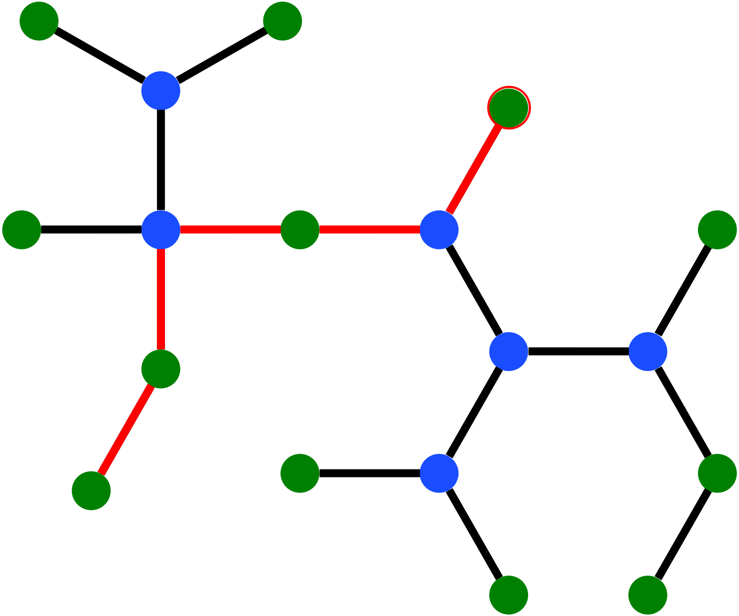

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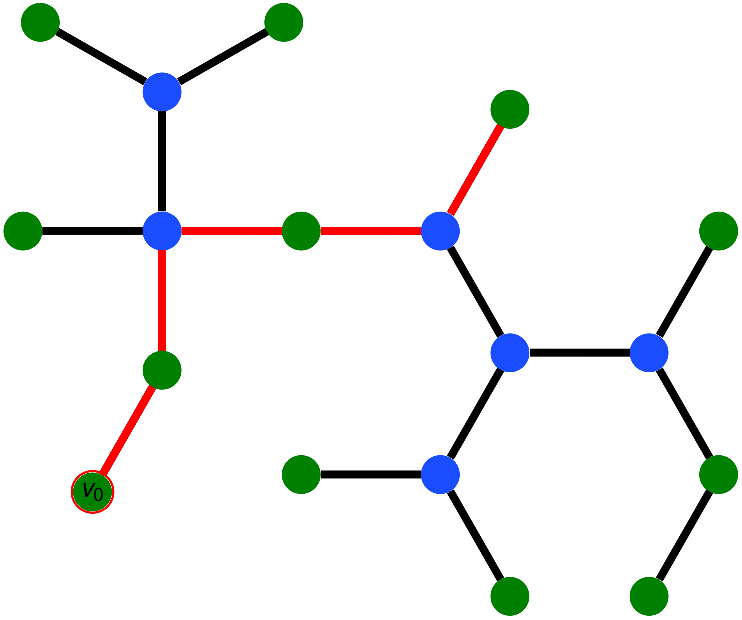

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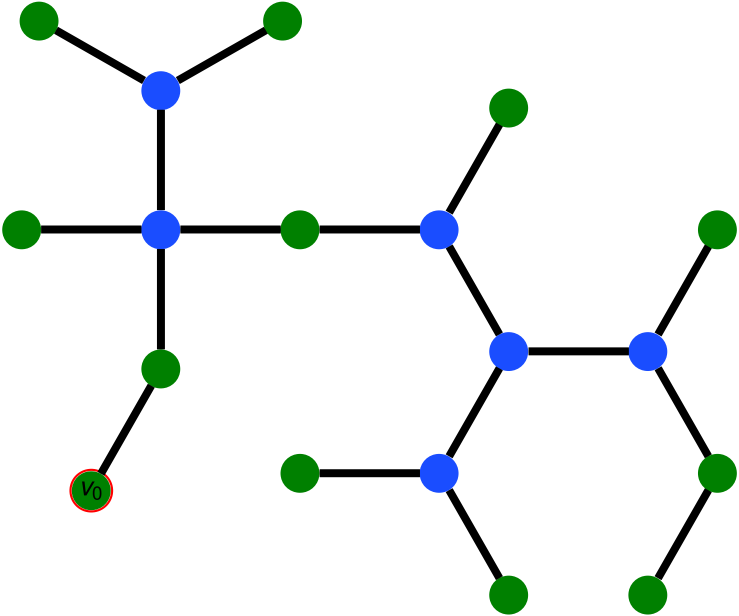

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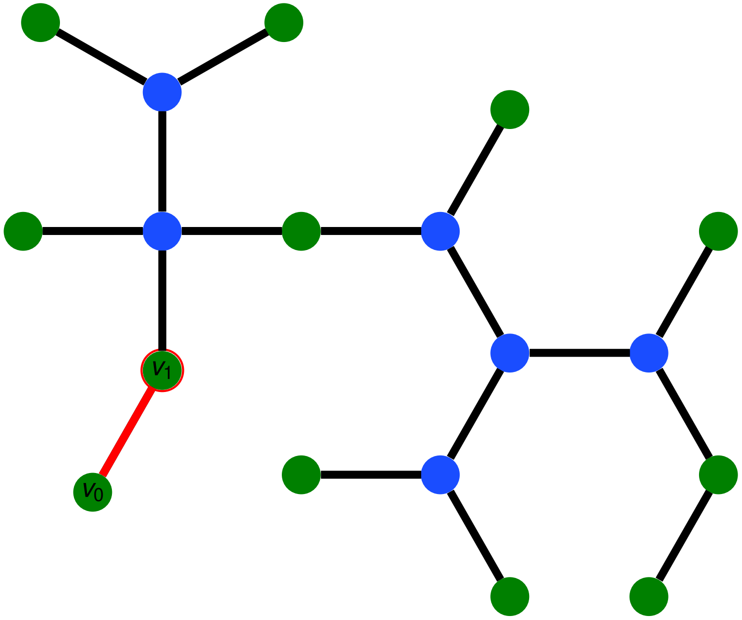

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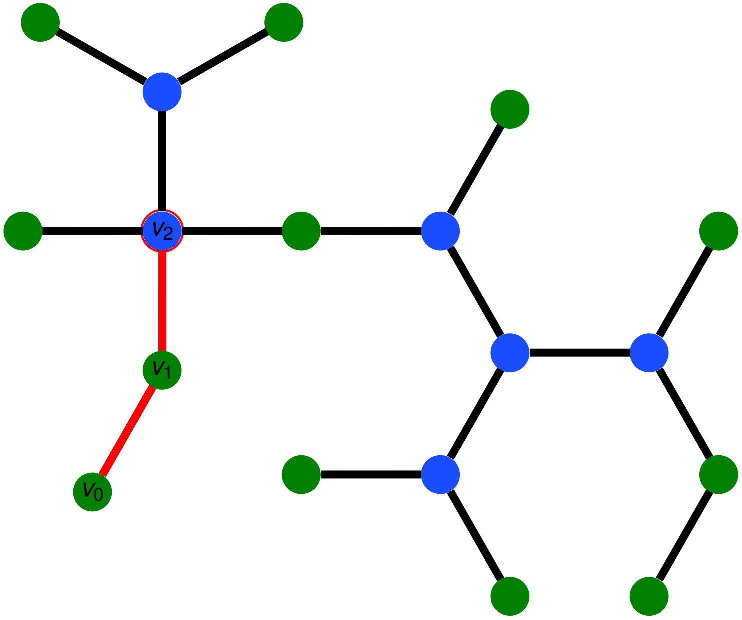

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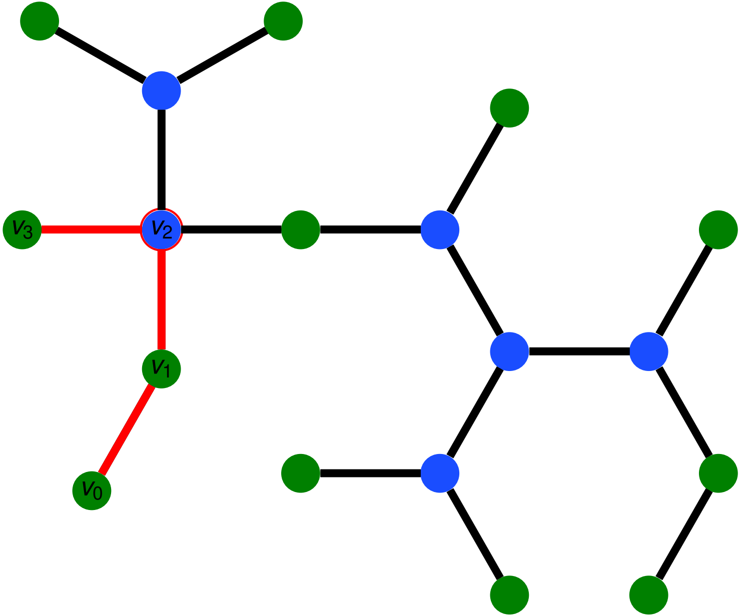

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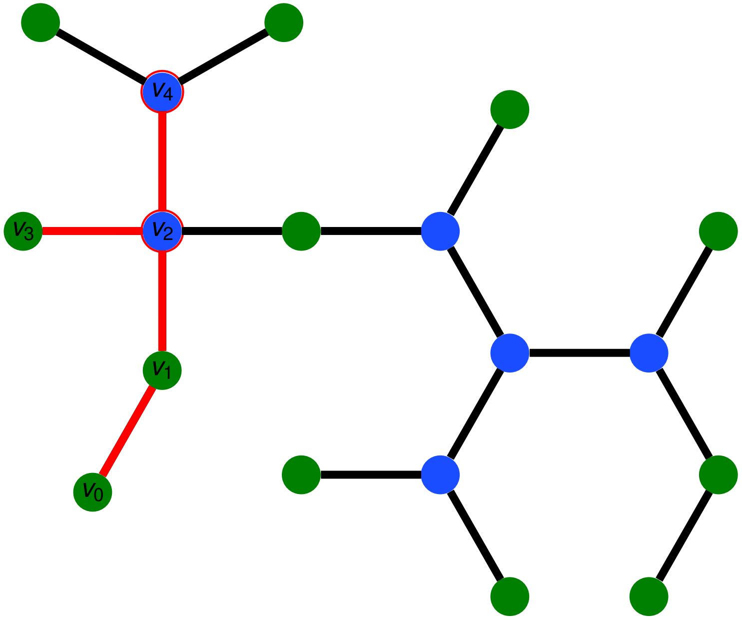

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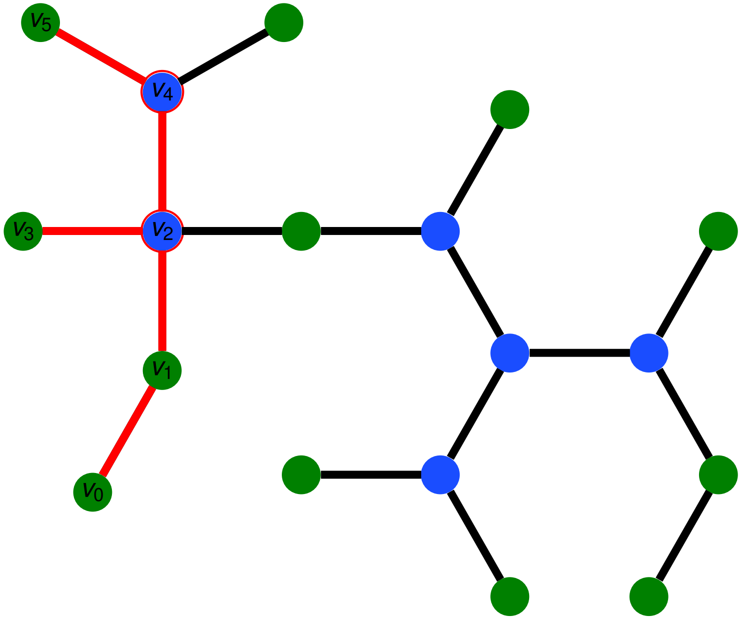

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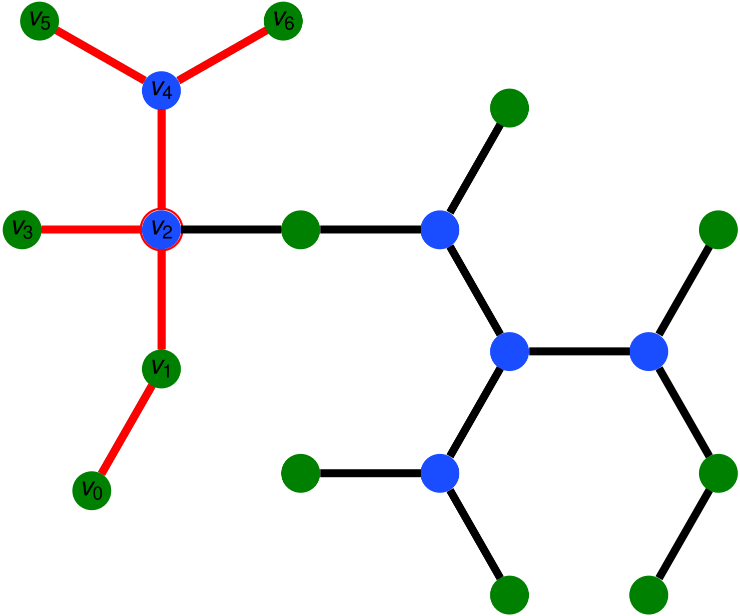

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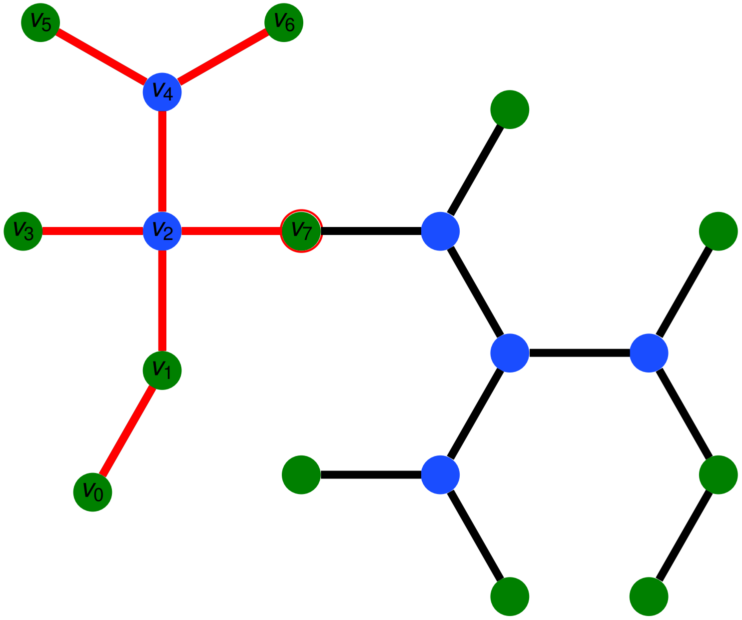

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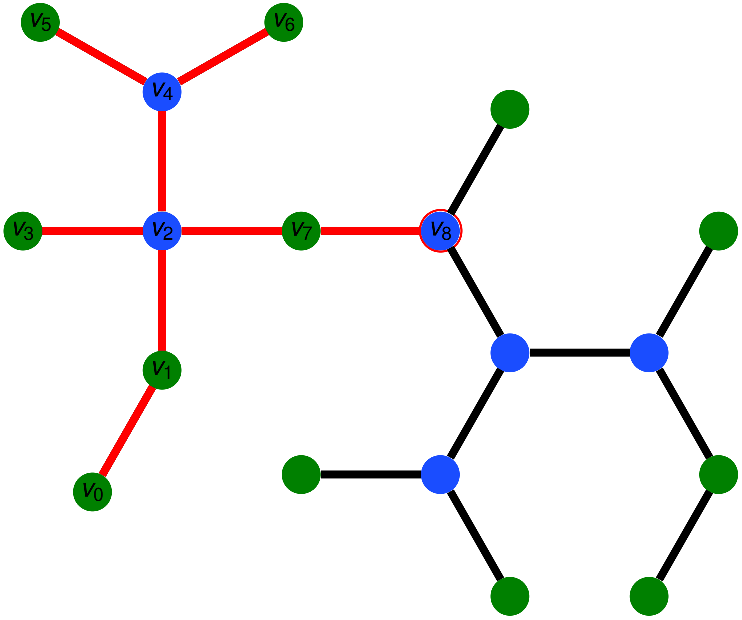

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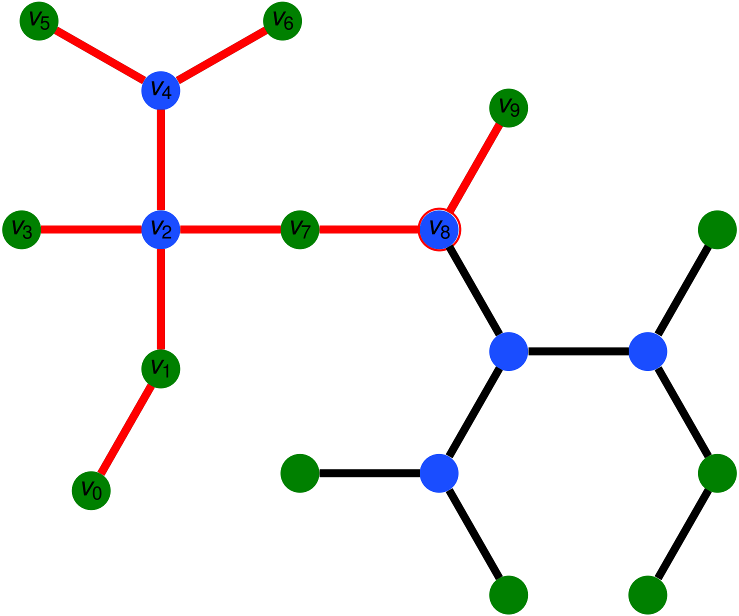

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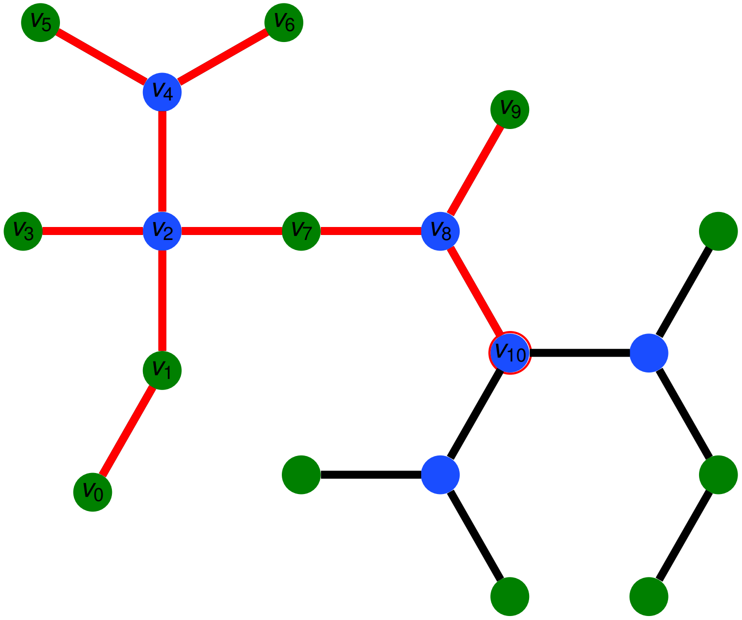

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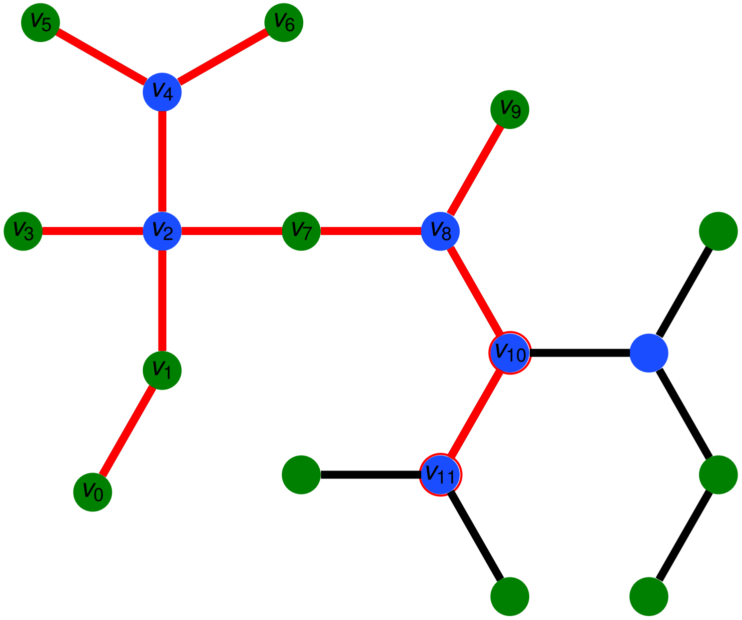

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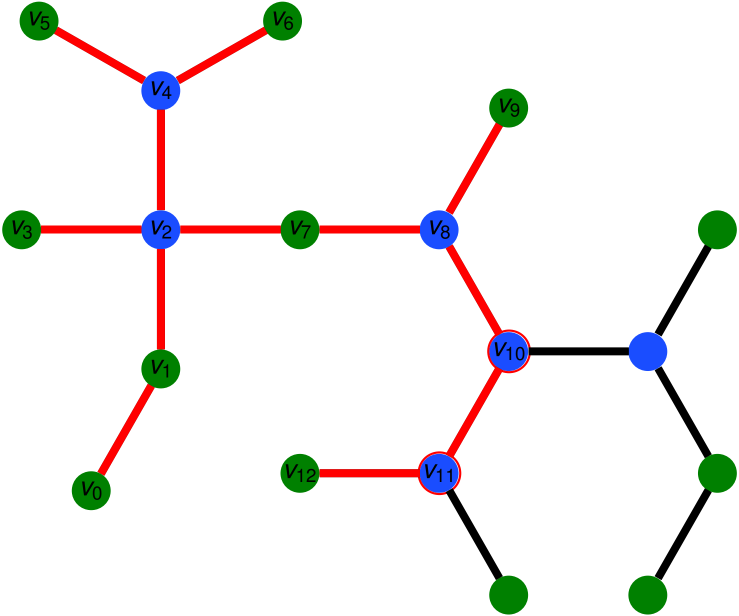

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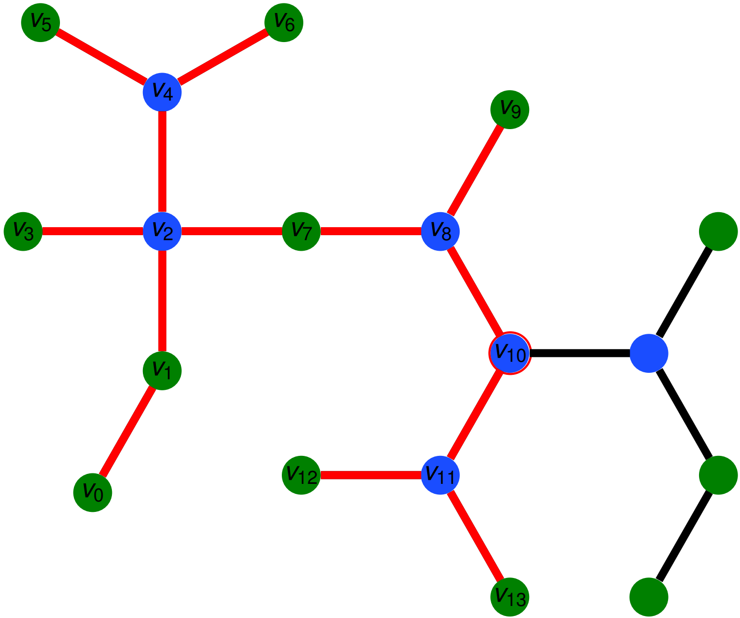

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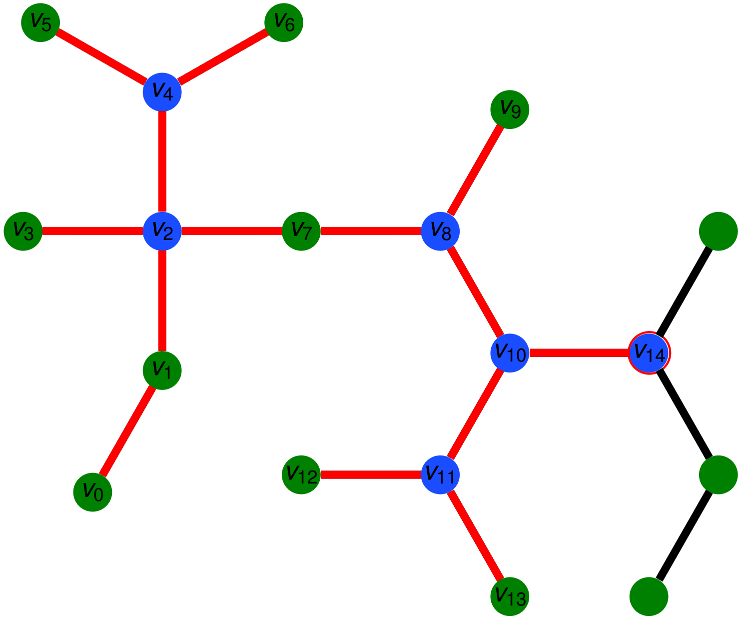

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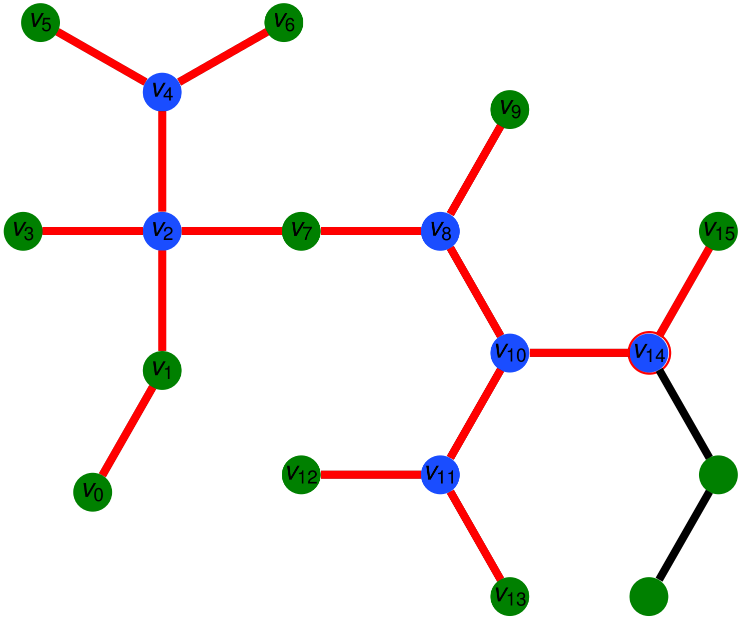

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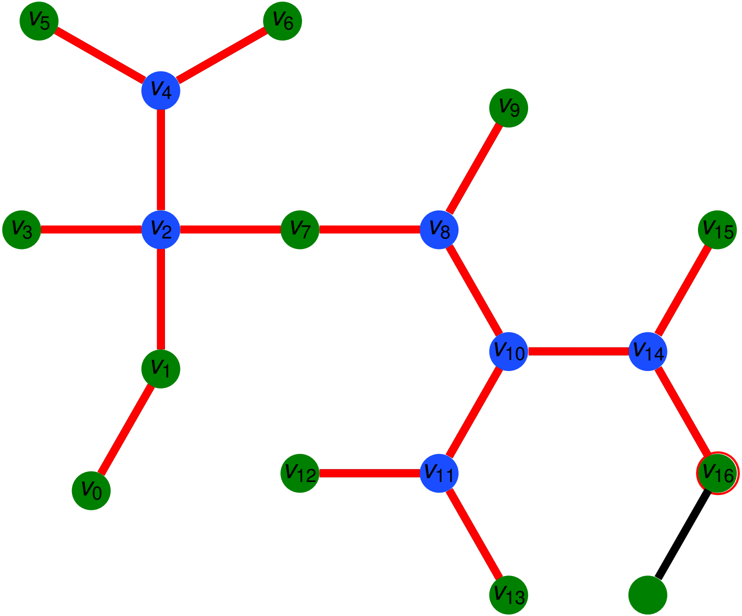

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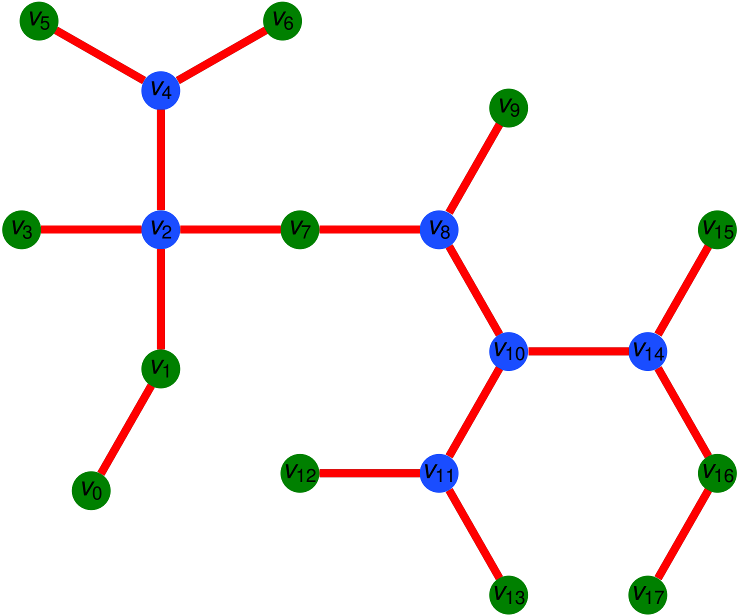

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We have to show that the number of boundary points is at most $\beta = 1 + \lfloor \log_2(k/3) \rfloor$ at any stage.

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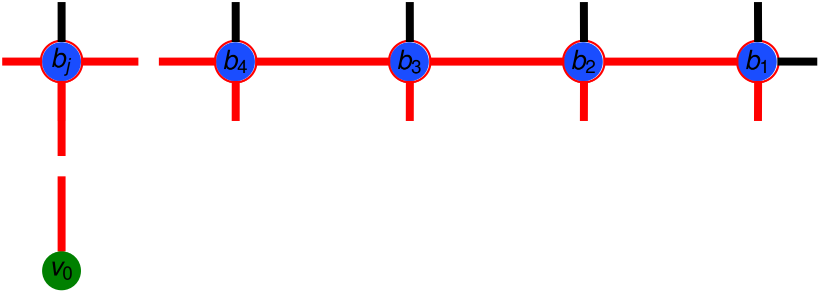

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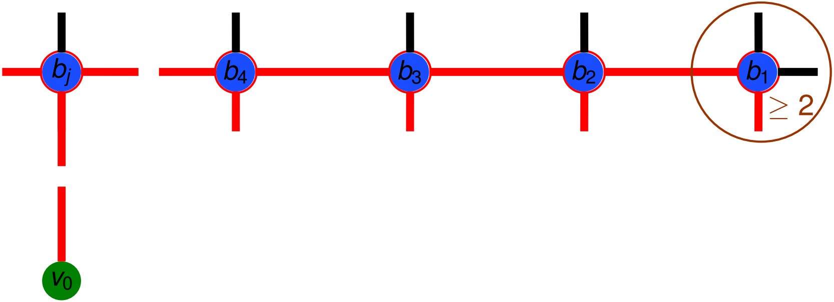

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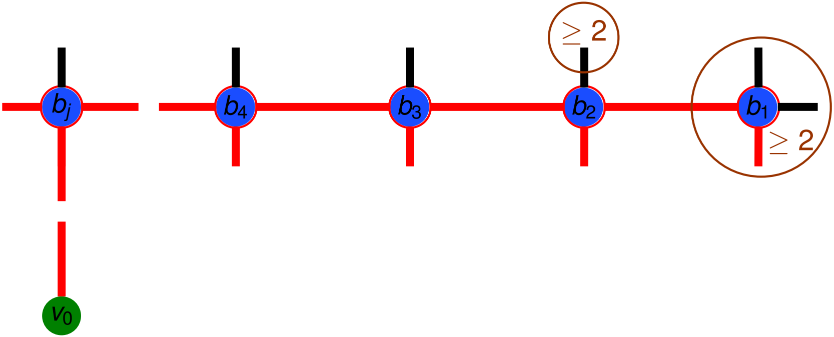

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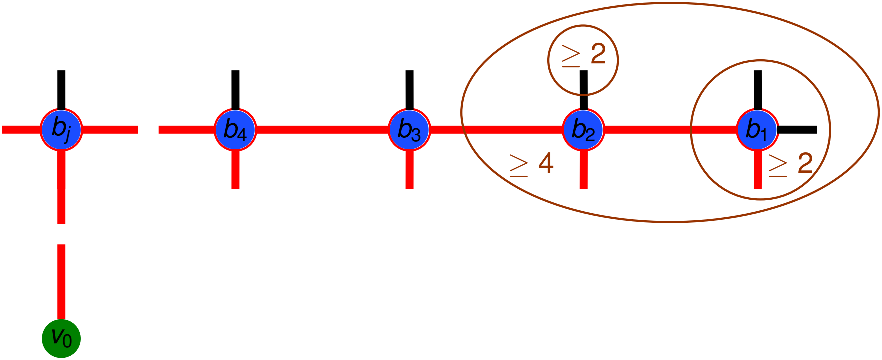

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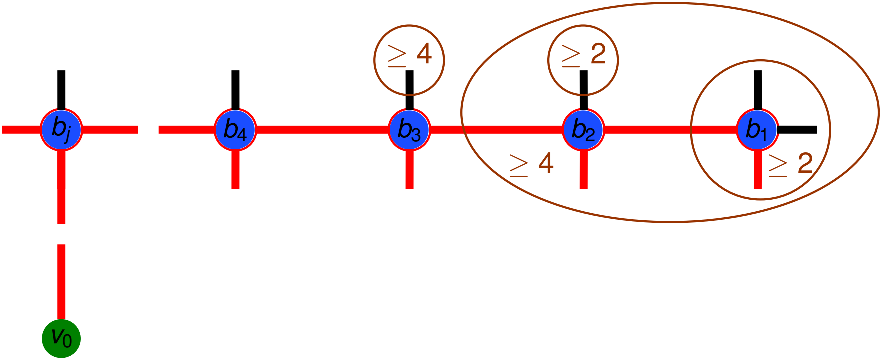

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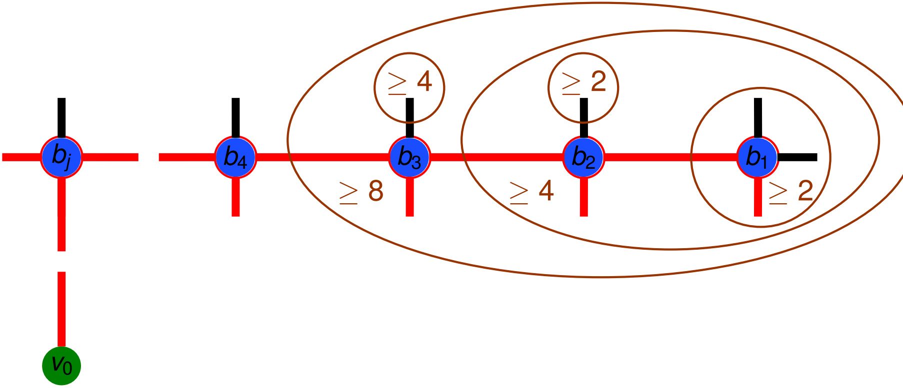

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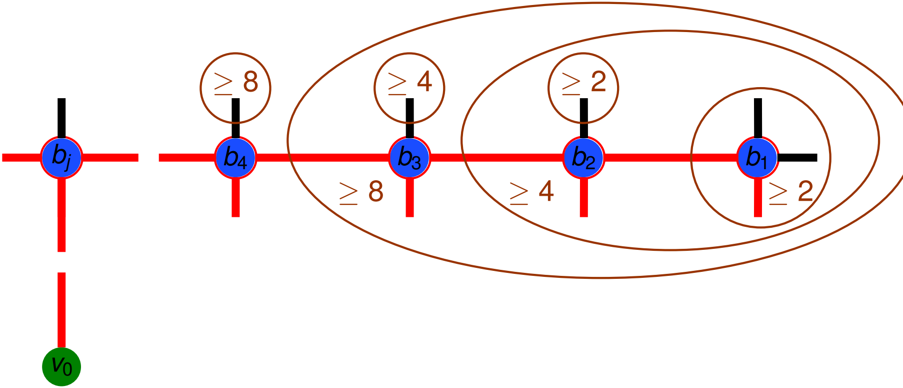

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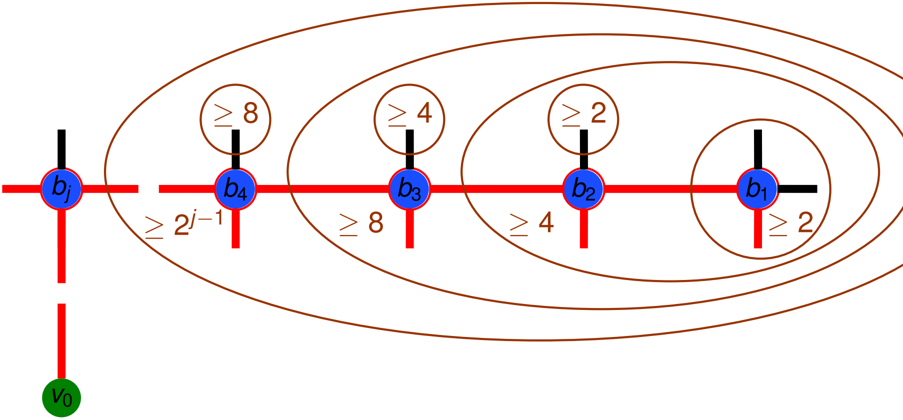

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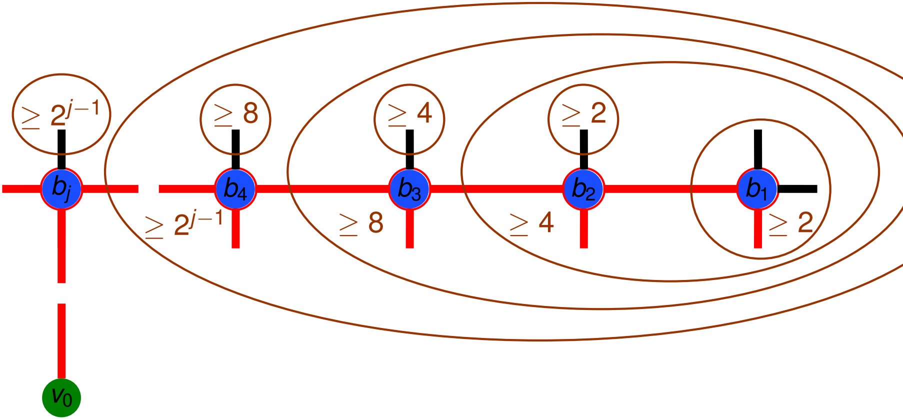

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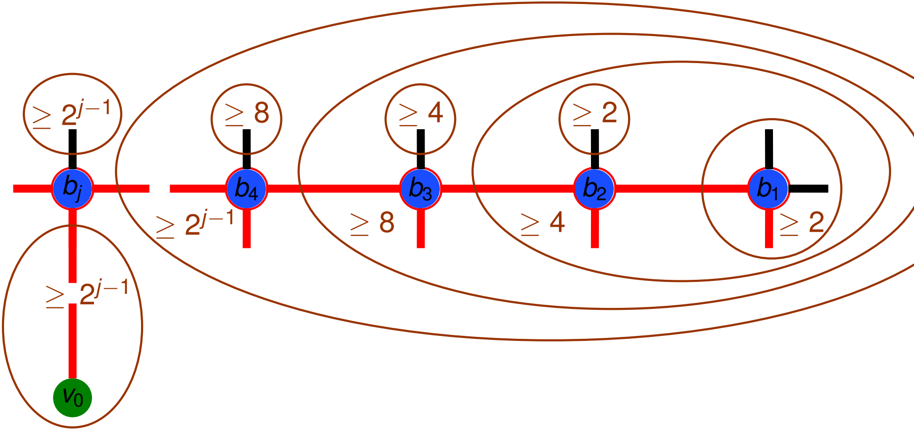
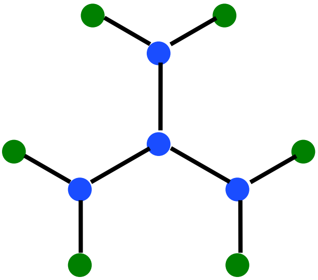


$\Rightarrow 3 \cdot 2^{j-1} \leq k \qquad \Rightarrow \qquad j \leq \beta.$ □

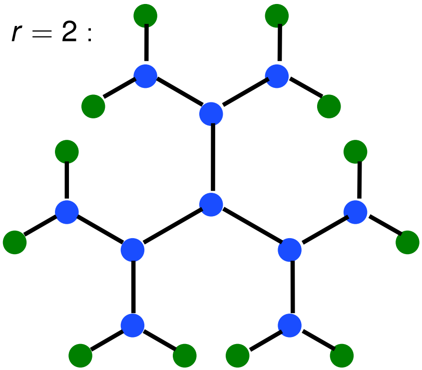
The theorem is best possible

In general we need $1 + \lfloor \log_2(k/3) \rfloor$ boundary points:

$r = 1$:



$r = 2$:



$3 \cdot 2^r$ terminals; need $r + 1$ boundary vertices

Recursion formulas (1)

For $X \subseteq T$, $X \neq \emptyset$, and $B \subseteq V(G) \setminus T$, $|B| \leq \beta$:

$$\sigma(X, B, 1) := \begin{cases} 0 & B = \emptyset \\ \infty & B \neq \emptyset \end{cases} \qquad \text{if } |X| = 1,$$

$$\sigma(X, B, 1) := \min \Big\{ \min\{\sigma(X \setminus \{x\}, B, i) + c(u, x) : x \in X,\ u \in (X \setminus \{x\}) \cup B,\ i \in \{1, 2\}\},$$
$$\min\{\sigma(X \setminus \{x\}, B \cup \{u\}, 1) + c(u, x) : x \in X,\ u \in V(G) \setminus (X \cup B)\} \Big\}$$
$$\text{if } |X| \geq 2 \text{ and } |B| < \beta,$$

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Recursion formulas (2)

For $X \subseteq T$, $X \neq \emptyset$, and $B \subseteq V(G) \setminus T$, $|B| \leq \beta$:

$$\sigma(X, B, 2) := \infty \qquad \text{if } B = \emptyset,$$

$$\sigma(X, B, 2) := \min\Big\{ \min\{\sigma(X, B\setminus\{b\}, i) + c(u, b) : b \in B,\ u \in X \cup (B\setminus\{b\}),\ i \in \{1, 2\}\},\ \min\{\sigma(X, B\setminus\{b\} \cup \{u\}, 1) + c(u, b) : b \in B,\ u \in V(G)\setminus(T \cup B)\} \Big\} \qquad \text{if } B \neq \emptyset.$$

Theorem

Let G be a complete graph, let $c : E(G) \to \mathbb{R}_+$ satisfy the triangle inequality. Let $T \subseteq V(G)$, $k := |T| \geq 3$, and $\beta := 1 + \lfloor \log_2(k/3) \rfloor$. Then $\sigma(T, \emptyset, 1)$ is the minimum length of a Steiner tree for T.

Algorithm

1. Compute the metric closure $(\bar{G}, \bar{c})$ of (G, c).
2. Compute the values $\sigma(X, B, i)$ with respect to $(\bar{G}, \bar{c})$.
3. Collect the edges of an optimum Steiner tree for T in $(\bar{G}, \bar{c})$.
4. Replace each of these edges by a shortest path in (G, c).

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Step 2 dominates the running time:

$$O\left(\sum_{i=1}^{k}\binom{k}{i}\sum_{j=0}^{\beta}\binom{n-k}{j}(in+jn)\right) = O\left(nk2^k n^\beta\right).$$

Now use $\beta = 1 + \lfloor \log_2(k/3) \rfloor \leq \log_2 k$.

Main result

Theorem

Our algorithm computes a minimum weight Steiner tree for a terminal set T in a weighted graph (G, c) in

$$O\left(nk2^{k+(\log_2 k)(\log_2 n)}\right)$$

time, where $n = |V(G)|$ and $k = |T|$. □

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Remark

This is the fastest known algorithm if

$$2 \log n \, (\log \log n)^3 \; < \; k \; < \; (n - \log^2 n)/2.$$

Summary

- Erickson, Monma, Veinott [1987]: $O\left(3^k n + 2^k(m + n\log n)\right)$
  fastest if $k < 4\log n$
- Fuchs, Kern, Mölle, Richter, Rossmanith, Wang [2007]: $O\left(2^{k+(k/2)^{1/3}(\ln n)^{2/3}}\right)$
  fastest if
  $4\log n < k < 2\log n(\log\log n)^3$
- here: $O\left(nk2^{k+(\log_2 k)(\log_2 n)}\right)$
  fastest if
  $2\log n(\log\log n)^3 < k < (n - \log^2 n)/2$
- enumeration: $O\left(m\alpha(m,n)2^{n-k}\right)$
  fastest if $k > (n - \log^2 n)/2$