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TWO-CONNECTED SPANNING SUBGRAPHS WITH AT MOST $\frac{10}{7}$OPT EDGES*

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Abstract. We present a $\frac{10}{7}$-approximation algorithm for the minimum 2-vertex-connected spanning subgraph problem. Similarly to the work of Cheriyan, Sebő, and Szigeti for 2-edge-connected spanning subgraphs, our algorithm is based on computing a carefully designed ear-decomposition.

Key words. vertex-connectivity, approximation algorithm, ear-decomposition

AMS subject classifications. 05C40, 05C85

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1. Introduction.

The 2-vertex-connected spanning subgraph problem (2VC) is a fundamental problem in survivable network design. Given a 2-vertex-connected graph, it asks for a 2-vertex-connected spanning subgraph with as few edges as possible.

In this paper, we describe a $\frac{10}{7}$-approximation algorithm for this problem. Let $\mathrm{OPT}(G)$ denote the minimum number of edges in any 2-connected spanning subgraph of G. Then our algorithm computes a 2-connected spanning subgraph of G with at most $\frac{10}{7}\mathrm{OPT}(G)$ edges in $O(n^3)$ time, where $n = |V(G)|$.

Previous work. 2VC is NP-hard because a graph G is Hamiltonian if and only if it contains a 2-vertex-connected spanning subgraph with n edges. Czumaj and Lingas [4] showed that the problem is APX-hard. Obtaining a 2-approximation algorithm is easy; for example, computing any open ear-decomposition and deleting trivial ears (cf. section 2) does the job.

Khuller and Vishkin [10] found a $\frac{5}{3}$-approximation algorithm. Garg, Vempala and Singla [6] improved the approximation ratio to $\frac{3}{2}$. The idea of both algorithms is to begin with a depth first search tree, add edges to obtain 2-vertex-connectivity, and delete edges that are not needed. There are also other $\frac{3}{2}$-approximation algorithms, e.g., by Cheriyan and Thurimella [2], who compute a smallest spanning subgraph with minimum degree 1 and extend it by a minimal set of edges to be 2-vertex-connected.

Better approximation ratios have been claimed several times: ratio $\frac{4}{3}$ by Vempala and Vetta [13], ratio $\frac{5}{4}$ by Jothi, Raghavachari, and Varadarajan [9], and ratio $\frac{9}{7}$ by Gubbala and Raghavachari [8]. However, we will prove that the approach of Vempala and Vetta [13] does not work (see the appendix), and Jothi, Raghavachari, and Varadarajan have withdrawn their claim (see [8]). Gubbala and Raghavachari [8] announced a full paper with a complete proof, but today there is only the approximately 70-page proof in Guballa's thesis [7] which contains some inconsistencies. According to Raghavachari (personal communication, 2016), they are no longer planning to revise their proof.

Apparently, the naturally arising question of whether there is an approximation algorithm with ratio better than $\frac{3}{2}$ for 2VC has been open for almost 25 years. It is also

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mentioned at the end of Nutov's recent survey [11], which discusses generalizations such as the min-cost version and k-connected subgraphs. Here we answer this question affirmatively.

Our approach. Our work is inspired by the work of Cheriyan, Sebő, and Szigeti [1], who proved the first ratio better than $\frac{3}{2}$ for the related 2-edge-connected spanning subgraph problem (2EC). Our algorithm works as follows:

1. We first delete some edges that we can identify as redundant (section 3).
2. Then we compute an open ear-decomposition with special properties (sections 4–6).
3. Finally we delete the "trivial" ears (those that consist of a single edge).

We compare our result to three lower bounds (section 7), two of which are well-known lower bounds even for 2EC. The third lower bound is new and exploits the properties obtained in steps 1 and 2.

2. Ear-decompositions.

In this paper, all graphs are simple and undirected. When we say 2-connected, we mean 2-vertex-connected. By $\delta(v)$ and $\Gamma(v)$ we denote the set of incident edges and the set of neighbors of a vertex v, respectively; so $|\delta(v)| = |\Gamma(v)|$ is the degree of v. When we write $G - e$, $G + f$, or $G - v$, we mean deleting an edge e, adding an edge f, or deleting a vertex v and all its incident edges. By $V(G)$ and $E(G)$ we denote the vertex set and edge set of G; let $n := |V(G)|$ be the number of vertices of the given graph G. Obviously $\mathrm{OPT}(G) \geq n$.

The following well-known graph-theoretic concept is due to Whitney [14].

DEFINITION 1. *An* ear-decomposition *of a graph G is a sequence $P_0, P_1, \ldots, P_k$ of graphs, where P_0 consists of a single vertex, $V(G) = V(P_0) \cup \cdots \cup V(P_k)$, $E(G) = E(P_1) \cup \cdots \cup E(P_k)$, and for all $i \in \{1, \ldots, k\}$ we have*

- *P_i is a circuit with exactly one vertex in $V(P_0) \cup \cdots \cup V(P_{i-1})$* (closed ear), *or*
- *P_i is a path whose endpoints, but no inner vertices, belong to $V(P_0) \cup \cdots \cup V(P_{i-1})$* (open ear).

A vertex in $V(P_i) \cap (V(P_0) \cup \cdots \cup V(P_{i-1}))$ is called an endpoint *of P_i (even if P_i is closed). An ear has one or two endpoints; its other vertices are called* inner *vertices. Let* $\mathrm{in}(P)$ *denote the set of inner vertices of an ear P. If P and Q are ears and $p \in \mathrm{in}(P)$ is an endpoint of Q, then Q is* attached *to P (at p).*

An ear-decomposition is open *if all ears except P_1 are open. We call an ear of length l (that is, with l edges) an* l-ear. *1-ears are also called* trivial *ears. An ear is called* pendant *if no nontrivial ear is attached to it.*

We can always assume that trivial ears come at the end of the ear-decomposition. Whitney [14] showed that a graph is 2-connected if and only if it has an open ear-decomposition (and a graph is 2-edge-connected if and only if it has an ear-decomposition). Therefore, 2VC is equivalent to computing an open ear-decomposition with as many trivial ears as possible (deleting the trivial ears yields a 2-connected spanning subgraph).

An arbitrary open ear-decomposition yields a 2-approximation algorithm because the number of edges in nontrivial ears is at most $2(n-1)$: for every nontrivial ear, the number of edges is at most twice the number of inner vertices. This is tight for 2-ears, and it already shows that we need to pay special attention to 2-ears and 3-ears. Let $\varphi(G)$ denote the minimum number of even ears in any ear-decomposition of G. The following result, based on a fundamental theorem of Frank [5], helps us in dealing with 2-ears.

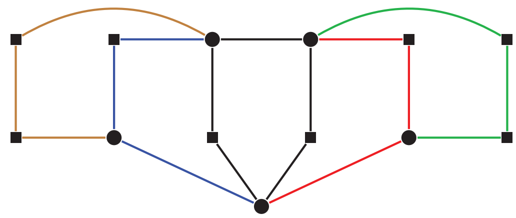

Fig. 1. *A graph G with $\varphi(G) = 0$ in which every open ear-decomposition has even ears or nonpendant 3-ears.*

Proposition 2 (Cheriyan, Sebő, and Szigeti [1]). *For any 2-connected graph G one can compute an open ear-decomposition with $\varphi(G)$ even ears in $O(n^3)$ time.*

Taking such an ear-decomposition and deleting all trivial ears yields a $\frac{3}{2}$-approximation for 2VC: for every nontrivial ear, the number of edges is at most $\frac{3}{2}$ times the number of inner vertices, except for 2-ears, for which we have to add $\frac{1}{2}$; hence we end up with at most $\frac{3}{2}(n-1) + \frac{1}{2}\varphi(G)$ edges, which is no more than $\frac{3}{2}\mathrm{OPT}(G)$ because $n - 1 + \varphi(G)$ is a lower bound (even for 2EC; see section 7).

To improve upon $\frac{3}{2}$, we have to look at 3-ears. Cheriyan, Sebő, and Szigeti [1] proceed by making all 3-ears pendant and such that inner vertices of different 3-ears are not adjacent; they show that this yields a $\frac{17}{12}$-approximation for 2EC (this ratio was later improved to $\frac{4}{3}$ in [12]). However, their ear-decomposition is not open, so we cannot use it for 2VC. In fact, there are 2-connected graphs that do not have an open ear-decomposition with $\varphi(G)$ even ears in which all 3-ears are pendant: see the example in Figure 1. Therefore we will require different properties for 3-ears in our ear-decomposition.

3. Redundant edges.

We first need to delete some edges of our graph.

Definition 3. *Let G be a 2-connected graph. An edge $e \in E(G)$ is* redundant *if $G - e$ is 2-connected and $\mathrm{OPT}(G - e) = \mathrm{OPT}(G)$.*

In other words, an edge is redundant unless it is contained in every optimum solution. Edges incident to a vertex of degree 2 are never redundant. The graph in Figure 1 has no redundant edges.

Of course it is in general difficult to decide whether a certain edge is redundant. But in some situations we can do it, as we show now. A slightly weaker version of the following lemma ($G - f$ 2-connected $\Rightarrow$ f redundant) was already shown by Chong and Lam [3].

Lemma 4. *Let G be a 2-connected graph, and let $a, b, c, d, e \in V(G)$ be five vertices with $\Gamma(a) = \{c, d\}$, $\Gamma(b) = \{c, e\}$, and $f := \{d, e\} \in E(G)$ (see Figure 2(a)). Then f is redundant if and only if $(G - c) - f$ is connected.*

Proof. If f is redundant, then $G - f$ is 2-connected, so $(G - c) - f = (G - f) - c$ is connected.

For the other direction, let $(G - c) - f$ be connected. Let H be a 2-connected spanning subgraph of G with minimum number of edges. If $f \notin E(H)$, we are done.

So assume $f \in E(H)$; then H contains all five edges of Figure 2(a), because the edges incident to the degree-2 vertices a and b belong to every 2-connected spanning subgraph. Moreover, $H - f$ is not 2-connected (due to the minimality of H), so let x be a vertex such that $(H - f) - x$ is disconnected. Clearly $x \in \{a, b, c\}$ because $H - x$ is connected, but deleting f from $H - x$ disconnects d from e. Then $(H - f) - c$ is disconnected.

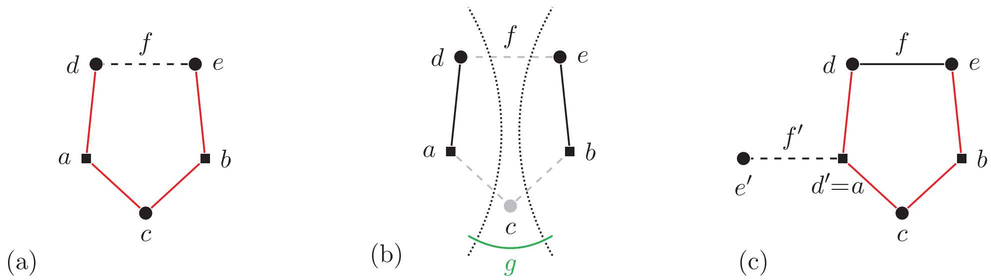

Fig. 2. *Identifying some redundant edges. Here and in all following figures, vertices shown as squares have no incident edges other than those shown. In* (a)*, f is redundant by Lemma* 4*. In* (b)*, g is an edge connecting the two connected components of* $(H-f)-c$*. In* (c)*, deleting f' identifies f as redundant.*

$(H-f)-c$ has exactly two connected components because $H-c$ is connected. Since $(G-c)-f$ is connected, there exists an edge $g \in E(G)\setminus\{f\}$ that connects the two connected components of $(H-f)-c$ (see Figure 2(b)). We claim that $H' := (H-f)+g$ is 2-connected, implying that f is redundant.

Suppose H' is not 2-connected, and let x be a vertex such that $H'-x$ is disconnected. As above, we have $x \in \{a,b,c\}$ and conclude that $H'-c$ is disconnected. But this contradicts the choice of g. ◻

In what follows we will often need the following property.

Property (P). *A graph G satisfies property* (P) *if it is* 2*-connected and for any five vertices $a,b,c,d,e \in V(G)$ with $\Gamma(a)=\{c,d\}$, $\Gamma(b)=\{c,e\}$, and $f=\{d,e\} \in E(G)$ we have that f is not redundant.*

We can obtain property (P) by successively deleting edges that we can identify as redundant by Lemma 4.

Corollary 5. *Let G be a* 2*-connected graph. Then one can compute a spanning subgraph $\bar{G}$ of G in $O(n^3)$ time such that* $\mathrm{OPT}(G)=\mathrm{OPT}(\bar{G})$ *and $\bar{G}$ satisfies property* (P)*.*

Proof. Identifying all sets of five vertices a,b,c,d,e with $\Gamma(a)=\{c,d\}$, $\Gamma(b)=\{c,e\}$, and $f=\{d,e\}\in E(G)$ can be done in $O(n^2)$ time by enumerating all pairs of vertices of degree 2. Let L be the set of such edges f.

Compute an open ear-decomposition of G. All edges of L that are trivial ears are redundant and can be removed. We now scan the $O(n)$ remaining edges in L one by one. For each of them, we can check in $O(|E(G)|)$ time by Lemma 4 whether it is redundant and, if so, delete it.

We claim that the resulting graph $\bar{G}$ satisfies property (P). Suppose not. Let a,b,c,d,e be five vertices violating property (P). Then $|\Gamma_G(a)|>2$ or $|\Gamma_G(b)|>2$ because otherwise $f=\{d,e\}$ would have been in L and hence been deleted. Consider the subgraph G' immediately before the last time that an edge incident to a or b was deleted. Say an edge $f'=\{d',e'\}$ was deleted with $d'=a$ (see Figure 2(c)). Then in particular there was a vertex a' with $\Gamma(a')=\{c',d'\}$. Then a' is a neighbor of $d'=a$ that is different from e', so $a'=d$ or $a'=c$. But note that $a'=d$ is impossible because if d has degree 2, then $f=\{d,e\}$ cannot be redundant. So $a'=c$, and hence $c'=b$ and (as b has degree 2 by the choice of f') $b'=e$. Again this is impossible because if e has degree 2, then $f=\{d,e\}$ cannot be redundant. ◻

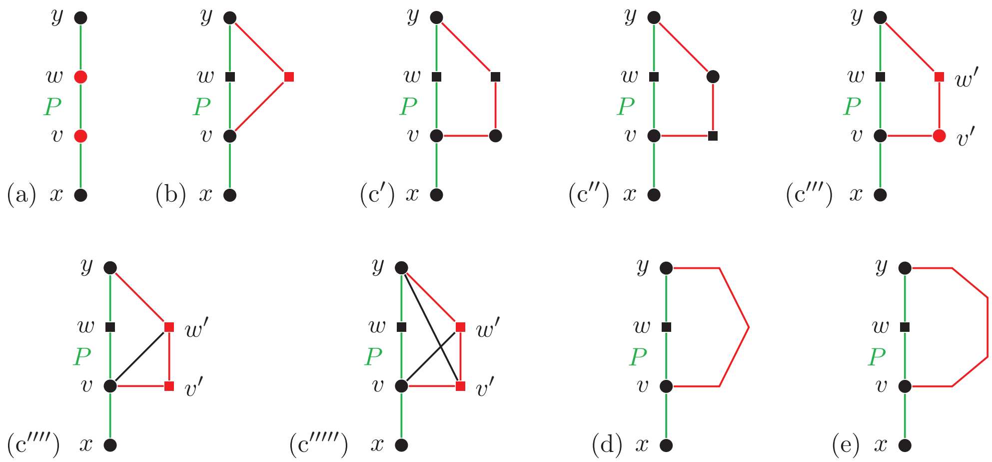

FIG. 3. *Possible 3-ears in an ear-decomposition with properties* (E1)–(E7): (a) *P is pendant; otherwise the first ear Q (red) attached to P (green) is* (b) *a 2-ear with inner vertex of degree* 2*;* (c′) *and* (c″) *a nonpendant 3-ear (which has a degree-*2 *inner vertex by* (E6)*);* (c‴), (c⁗)*, and* (c′′′′′) *a pendant 3-ear;* (d) *a 4-ear;* and (e) *an ear of length at least* 5*. Again, vertices shown as squares have no incident edges other than those shown. Inner vertices of pendant ears are shown in red.*

Hence, in the following, we may assume property (P).

4. A new ear-decomposition. For a graph G with property (P) we will compute an ear-decomposition with properties (E1)–(E7) below. The edges in the nontrivial ears of this ear-decomposition will constitute a $\frac{10}{7}$-approximation for 2VC. (E1) and (E3) are the same as in [1] (except that our ear-decomposition is open); (E4)–(E7) deal with nonpendant 3-ears.

(E1) The ear-decomposition is open and has $\varphi(G)$ even ears.

(E2) Pendant 3-ears come at the end of the ear-decomposition, followed only by trivial ears.

(E3) Inner vertices of different pendant 3-ears are not adjacent in G.

Moreover, for every nonpendant 3-ear P and the first nontrivial ear Q attached to P, say at v, where $E(P) = \{\{x, v\}, \{v, w\}, \{w, y\}\}$, we have the following:

(E4) The other endpoint of Q is y, and $y \neq x$.

(E5) If Q is a pendant 3-ear, and $E(Q) = \{\{v, v'\}, \{v', w'\}, \{w', y\}\}$, then w' has degree 2 or $(\Gamma(w') = \{y, v', v\}$ and $\Gamma(v') \subseteq \{v, w', y\})$.

(E6) The vertex w has degree 2 in G.

(E7) If Q is a 2-ear, then the inner vertex of Q has degree 2 in G.

Conditions (E4)–(E7) say that every 3-ear has one of the properties shown in Figure 3. The first ear can be a 3-ear only if it is pendant, i.e., only if $n = 3$.

In the following we show how to compute such an ear-decomposition (if the graph has property (P)). We get condition (E1) from Proposition 2 and condition (E2) by reordering the ears. We maintain these conditions throughout. In the next section we deal with conditions (E3)–(E5). If one of them is violated, we can modify the ear-decomposition and increase the number of trivial ears. Then, in section 6, we deal with the other conditions.

In any ear-decomposition there are fewer than n nontrivial ears, so we can check conditions (E4)–(E7) in $O(n)$ time, and condition (E3) in $O(n^2)$ time.

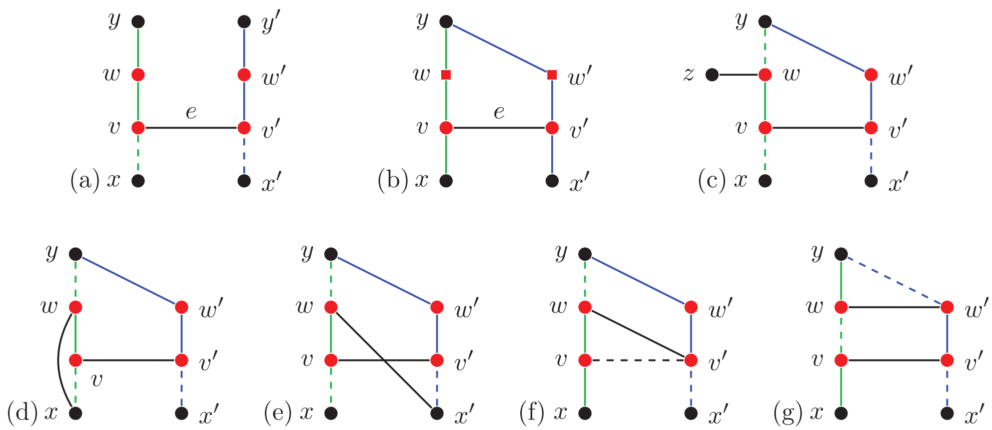

Fig. 4. *Proof of Lemma* 6. *Two adjacent pendant* 3*-ears* P_i *(green) and* P_j *(blue). Black edges are old trivial ears. Dashed edges form new trivial ears.*

5. How to obtain properties (E3), (E4), and (E5). We deal first with condition (E3).

Lemma 6. *Let* G *be a graph with property* (P)*, and let an ear-decomposition of* G *be given that satisfies* (E1) *and* (E2) *but not* (E3)*. Then we can compute in* $O(n^2)$ *time an ear-decomposition that satisfies* (E1) *and* (E2) *and has more trivial ears.*

Proof. Let P_i and P_j be pendant 3-ears with $E(P_i) = \{\{x,v\},\{v,w\},\{w,y\}\}$, $E(P_j) = \{\{x',v'\},\{v',w'\},\{w',y'\}\}$, and $e = \{v,v'\} \in E(G)$.

We consider several cases. In each case we construct a new open pendant 5-ear P' which can be added to the ear-decomposition just before the pendant 3-ears. P_i and P_j will be removed, and the number of trivial ears increases by 1.

Case 1: $y \neq y'$.

Let P' be the 5-ear consisting of the y-v-path in P_i, the edge $\{v,v'\}$, and the v'-y'-path in P_j (Figure 4(a)).

Case 2: $y = y'$ and $|\delta(w)| = |\delta(w')| = 2$.

As e is a trivial ear in the ear-decomposition, $G-e$ is 2-connected, and so $(G-e)-y$ is connected. By Lemma 4 this means that e is redundant, violating property (P). Therefore this case cannot happen (Figure 4(b)).

Case 3: $y = y'$ and $|\delta(w)| > 2$.

Case 3.1: There is a $z \in \Gamma(w)\backslash(V(P_i) \cup V(P_j))$.

Let P' be a 5-ear with edges $\{z,w\},\{w,v\},\{v,v'\},\{v',w'\}$, and $\{w',y\}$ (Figure 4(c)).

Case 3.2: $x \in \Gamma(w)$.

Let P' be a 5-ear with edges $\{x,w\},\{w,v\},\{v,v'\},\{v',w'\}$, and $\{w',y\}$ (Figure 4(d)).

Case 3.3: $x' \in \Gamma(w)$.

Let P' be a 5-ear with edges $\{x',w\},\{w,v\},\{v,v'\},\{v',w'\}$, and $\{w',y\}$ (Figure 4(e)).

Case 3.4: $v' \in \Gamma(w)$.

Let P' be a 5-ear with edges $\{x,v\},\{v,w\},\{w,v'\},\{v',w'\}$, and $\{w',y\}$ (Figure 4(f)).

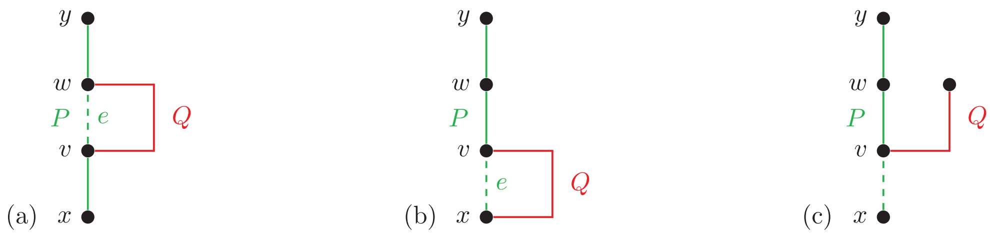

Fig. 5. *Proof of Lemma* 7. *The nonpendant* 3*-ear* P *(green) and the first ear* Q *(red) attached to it violate condition* (E4). *The dashed edge becomes a trivial ear in the new ear-decomposition.*

Case 3.5: $w' \in \Gamma(w)$.

Let P' be a 5-ear with edges $\{x, v\}, \{v, v'\}, \{v', w'\}, \{w', w\}$, and $\{w, y\}$ (Figure 4(g)).

Case 4: $y = y'$ and $|\delta(w')| > 2$.

This case is symmetric to Case 3. ◻

Now we consider condition (E4).

Lemma 7. *Given an ear-decomposition of* G *that satisfies* (E1) *and* (E2) *but not* (E4)*, we can compute in* $O(n)$ *time an ear-decomposition that satisfies* (E1) *and* (E2) *and has more trivial ears.*

Proof. Let P be a nonpendant 3-ear with $E(P) = \{\{x, v\}, \{v, w\}, \{w, y\}\}$ such that the first nontrivial ear Q attached to P has v as an endpoint, but the other endpoint of Q is not y.

If the other endpoint of Q is w or x (this will always be the case if P is the first (closed) ear), let e be the edge of P that connects the two endpoints of Q. We can modify P by replacing e by Q. The new ear is even if and only if Q is. The edge e becomes a new trivial ear, and Q vanishes (Figures 5(a) and 5(b)).

If the other endpoint of Q does not belong to $V(P)$, we can replace P and Q by a new ear consisting of Q and the v-y-path in P. This new ear is even if and only if Q is, and can be put at the position of Q in the ear-decomposition. The edge $\{v, x\}$ becomes a new trivial ear (Figure 5(c)). ◻

Now we turn to (E5).

Lemma 8. *Let* G *be a graph with property* (P)*, and let an ear-decomposition of* G *be given that satisfies* (E1)–(E4) *but not* (E5). *Then we can compute in* $O(n)$ *time an ear-decomposition that satisfies* (E1) *and* (E2) *and has more trivial ears.*

Proof. Let P be a nonpendant 3-ear with $E(P) = \{\{x, v\}, \{v, w\}, \{w, y\}\}$, and let Q be the first nontrivial ear attached to it. Suppose Q is a pendant 3-ear. By (E3) it has endpoints v and y, so let $E(Q) = \{\{v, v'\}, \{v', w'\}, \{w', y\}\}$. By (E2), only pendant 3-ears and trivial ears are attached to P. Therefore we may assume that after P there are only the pendant 3-ears and trivial ears in the ear-decomposition.

Suppose w' has degree greater than 2. Then property (E5) requires $\Gamma(w') = \{y, v', v\}$ and $\Gamma(v') \subseteq \{v, w', y\}$, so suppose this does not hold.

Case 1: There exists a $z \in \Gamma(w') \backslash \{y, v', v\}$.

Case 1.1: $z = w$.

Then let P' be a 5-ear with edges $\{x, v\}, \{v, v'\}, \{v', w'\}, \{w', w\}$, and $\{w, y\}$ (Figure 6(a)).

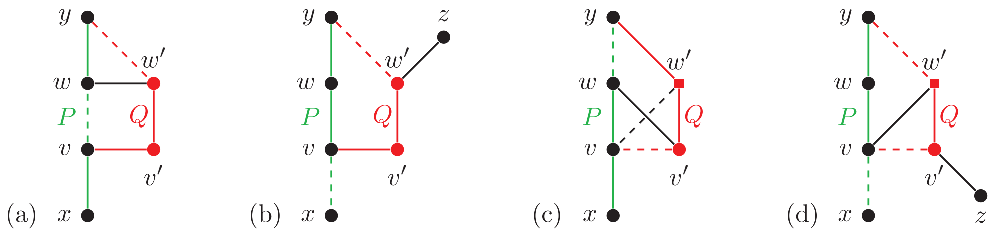

FIG. 6. *Proof of Lemma* 8. *The nonpendant* 3*-ear* P *(green) and the first ear* Q *(red) attached to it violate condition* (E5). Q *is a pendant* 3*-ear. Black edges are trivial ears. The dashed edges become trivial ears in the new ear-decomposition. Again, squares indicate vertices that have no incident edges other than those shown. In* (b) *and* (d), $z = x$ *is possible.*

Case 1.2: $z \neq w$.

Then let P' be a 5-ear with edges $\{y, w\}, \{w, v\}, \{v, v'\}, \{v', w'\}$, and $\{w', z\}$ (Figure 6(b)).

Case 2: $\Gamma(w') = \{y, v', v\}$, and there exists a $z \in \Gamma(v') \backslash \{v, w', y\}$.

Case 2.1: $z = w$.

Then let P' be a 5-ear with edges $\{x, v\}, \{v, w\}, \{w, v'\}, \{v', w'\}$, and $\{w', y\}$ (Figure 6(c)).

Case 2.2: $z \neq w$.

Then let P' be a 5-ear with edges $\{y, w\}, \{w, v\}, \{v, w'\}, \{w', v'\}$, and $\{v', z\}$ (Figure 6(d)).

We replace the 3-ears P and Q by the new open 5-ear P'. Due to condition (E3), z in Cases 1 and 2 cannot be an inner vertex of a pendant 3-ear, so we can put P' at the position of P in the ear-decomposition. In each case there is one trivial ear more than before. ◻

6. How to obtain properties (E6) and (E7).

In the following lemma we show how to obtain properties (E6) and (E7) simultaneously, maintaining (E1) and (E2). The number of trivial ears will not decrease.

LEMMA 9. *Given an ear-decomposition of* G *that satisfies* (E1) *and* (E2), *we can compute in* $O((j+1)n^2)$ *time an ear-decomposition that satisfies* (E1), (E2), (E6), *and* (E7), *and in which the number of trivial ears is* j *more than before, where* $j \geq 0$.

Proof. Let P_i be the first nonpendant 3-ear that violates at least one of the conditions (E6) and (E7). We will perform changes to the ear-decomposition, maintaining (E1) and (E2), such that afterwards the number of trivial ears increases or this number remains constant, but the first ear violating (E6) or (E7) has a larger index.

At the beginning and after every modification of the ear-decomposition we apply Lemma 7 in order to ensure that the ear-decomposition satisfies condition (E4), and, to ensure (E2), we move pendant 3-ears to the end of the ear-decomposition, followed only by the trivial ears.

Let Q be the first nontrivial ear attached to P_i. Let $E(P_i) = \{\{x, v\}, \{v, w\}, \{w, y\}\}$, where v and y are the endpoints of Q. We will proceed with the following steps.

Step 1: If P_i violates (E6), then w has degree greater than 2.

Let $X := \bigcup_{j=0}^{i} V(P_j)$. Let R be the first ear attached to P_i at w (possibly trivial) and u, the other endpoint of R. Consider the following procedure (see Figure 7(a) for an example):

$S := R$;
$a := u$;
while $a \notin X$ **do**
 let P' be the ear with $a \in \text{in}(P')$;
 let b be an endpoint of P', if possible such that the a-b-path T in P' has even length;
 $a := b$;
 $S := S \cup T$;

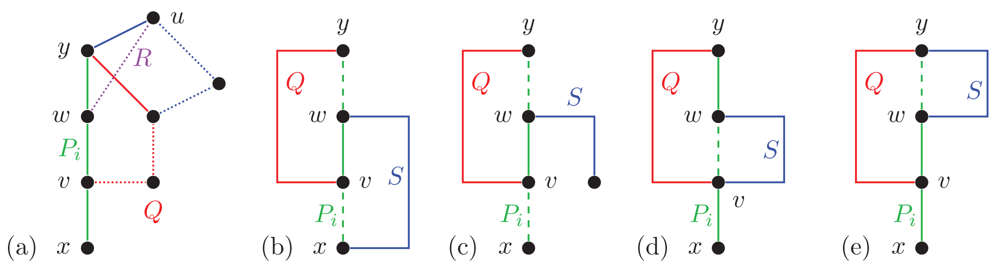

FIG. 7. *Proof of Lemma* 9: *Condition* (E6) *is violated.* P_i *(green) is a nonpendant* 3*-ear,* Q *(red) is the first ear attached to it (at* v*), and* R *is the first ear with endpoint* w*.* (a) *shows an example of the computation of* S *(dotted edges), with* R *in purple and another ear in blue. In* (b)–(e)*,* S *is shown in blue, and the resulting trivial ears are dashed.*

This procedure terminates because the index of P' decreases in every iteration. For the same reason, S is always a path; it ends in $X\backslash\{w\}$ because R is the first ear attached to P_i at w.

In the following we will create a new open ear that contains all of S. Every ear (except R) that was used for the construction of S, i.e., that played the role of P' in the above procedure, has a part in S and another part outside S. We will remove the edges in S from these ears. If these ears were odd, they remain odd. Note that Q can be one of these ears (then it is the last one), as in Figure 7(a).

Case 1: S ends in $X\backslash\{v, y\}$.

Then Q and S are disjoint. Replace P_i and Q by an ear that consists of S, $\{w, v\}$, and Q (see Figures 7(b) and 7(c)). Remove the edges in S from all other ears. The edges $\{y, w\}$ and $\{v, x\}$ become trivial ears. The total number of trivial ears has increased (we have at least two more and at most one less, namely R if it was trivial).

Case 2: S ends in v.

Then S has more than one edge. Replace P_i by an ear that consists of $\{x, v\}$, S, and $\{w, y\}$ (see Figure 7(d), but again note that S can contain part of Q as in Figure 7(a)). Remove the edges in S from all other ears. The edge $\{v, w\}$ becomes a trivial ear.

Case 3: S ends in y.

Then S again has more than one edge. Replace P_i by an ear that consists of $\{x, v\}$, $\{v, w\}$, and S (see Figure 7(e), but note that S can contain part of Q). Remove the edges in S from all other ears. The edge $\{w, y\}$ becomes a trivial ear.

Note that in each of the three cases the new ear is open and has more than three edges and thus does not violate condition (E2), (E6), or (E7). We can put it at the position of P_i.

The only possible new even ear is the one we designed from S. But then R is

FIG. 8. *Proof of Lemma* 9: *Condition* (E7) *is violated.* (a) P_i *(green) is a nonpendant* 3*-ear, and the* 2*-ear* Q *(red) is the first ear attached to it. Here* w *has degree* 2*, but* w' *has larger degree.* (b) *The new ears* P' *(green) and* Q' *(red).*

even (and vanishes), or one of the ears—part of which belongs to S—changes from even to odd, or Q is even and vanishes in Case 1. So (E1) is maintained.

The (possibly trivial) ear R vanishes, but at least one edge of P_i becomes a new trivial ear, so the number of trivial ears does not decrease.

If a new pendant 3-ear P_h with index $h < i$ arises (violating (E2)), then this means that an ear was attached to it that became trivial or vanished. If an existing ear becomes trivial, the number of trivial ears increases, and we start over again. The only ears that can vanish are R and Q. If R was attached to P_h with $h < i$, then $R = S$ and the new ear is also attached to P_h. If Q was attached to P_h, then so is the new ear (at y). Therefore the ears up to P_i do not need reordering for ensuring (E2). This implies that the number of trivial ears or the index of the first ear violating (E6) or (E7) increases.

We may also have created new 3-ears that violate condition (E6) or (E7), but they come later in the ear-decomposition.

Step 2: If P_i satisfies (E6) but violates (E7), then Q is a 2-ear with inner vertex w' of degree at least 3.

Replace P_i and Q by the ears P' and Q', where $E(P') = \{\{x, v\}, \{v, w'\}, \{w', y\}\}$ and $E(Q') = \{\{v, w\}, \{w, y\}\}$ (see Figure 8). This will violate condition (E6) for P'. Now apply Step 1 to P'.

We now consider the running time. Each of Steps 1 and 2 takes $O(n)$ time, including resorting new pendant 3-ears and trivial ears to the end. As soon as (E4) is violated for some ear, we apply Lemma 7, which increases the number of trivial ears. While this does not happen, after i iterations of Steps 1 and 2, the first i ears satisfy (E6) and (E7). So after at most $n - 1$ iterations we are either done or we increase the number of trivial ears. ∎

COROLLARY 10. *Given a graph* G *with property* (P)*, one can compute an ear-decomposition of* G *with properties* (E1)–(E7) *in* $O(n^3)$ *time.*

Proof. We first compute an ear-decomposition with property (E1), using Proposition 2. We obtain (E2) by reordering ears if necessary. Then we apply Lemmas 6–9 until all properties are satisfied. Lemma 9 does not decrease the number of trivial ears, and each application of Lemma 6, 7, or 8 increases the number of trivial ears. As the number of trivial ears can increase by at most $n - 2$ (the number of edges in nontrivial ears is always between n and $2n - 2$), the running time follows. ∎

7. Lower bounds.

The first two lower bounds are well known; they are lower bounds even for 2EC.

LEMMA 11 (Cheriyan, Sebő, and Szigeti [1]). *Let* G *be a* 2*-connected graph. Then*

$$\mathrm{OPT}(G) \geq n - 1 + \varphi(G).$$

Proof. Let H be a 2-connected spanning subgraph of G. Any ear-decomposition of H can be extended to an ear-decomposition of G by adding trivial ears, so H contains at least $\varphi(G)$ even ears. Consequently, H has at least $\varphi(G)$ ears, and hence $|E(H)| \geq n - 1 + \varphi(G)$. ∎

LEMMA 12 (Garg, Vempala, and Singla [6]). *Let G be a 2-connected graph, and let W be a proper subset of the vertices. Let q_W be the number of connected components of $G[W]$, the subgraph of G induced by W (and $q_\emptyset := 0$). Then*

$$\mathrm{OPT}(G) \geq |W| + q_W.$$

Proof. Let H be a 2-connected spanning subgraph of G. For each connected component C of $G[W]$ there are at least two edges in H that connect C with $V(G)\setminus W$. Moreover, every vertex in C has degree at least 2 in H, so H contains at least $|V(C)|+1$ edges with at least one endpoint in C. Summing over all C yields that $|E(H)| \geq |W| + q_W$. ∎

The following lower bound applies only to 2VC, and it requires property (P) and an ear-decomposition satisfying (E1)–(E7).

LEMMA 13. *Let G be a graph with property* (P) *and an ear-decomposition with* (E1)–(E7). *Let k be the number of nonpendant 3-ears P such that the first nontrivial ear attached to P has length 2 or 3. Then*

$$\mathrm{OPT}(G) \geq n - 1 + k.$$

Proof. Let H be a 2-connected spanning subgraph of G with minimum number of edges. We will remove k edges from H and still maintain a connected graph.

We scan the ears of our ear-decomposition of G in reverse order. If the current ear P is a nonpendant 3-ear such that the first nontrivial ear Q attached to P has length 2 or 3, we delete an edge from H that is chosen as follows.

Let $E(P) = \{\{x, v\}, \{v, w\}, \{w, y\}\}$, and let Q have endpoints v and y (cf. condition (E4)). In each case we show that H contains a cycle whose vertices all belong to P or Q, and we will delete an edge of Q from H.

Case 1: Q is a 2-ear.

Then w (by (E6)) and the inner vertex u of Q (by (E7)) have degree 2 in G. Therefore H contains all four edges $\{v, w\}, \{w, y\}, \{y, u\}$, and $\{u, v\}$. Deleting the edge $\{u, v\}$ will therefore not disconnect the graph (Figure 9(a)).

Case 2: Q is a nonpendant 3-ear.

Let $E(Q) = \{\{v, v'\}, \{v', w'\}, \{w', y\}\}$. By (E6), w has degree 2 in G. Since Q is also a nonpendant 3-ear, (E6) also implies that v' or w' has degree 2 in G. If w' has degree 2 in G (Figure 9(b)), then by property (P) the edge $\{v, v'\}$ is not redundant. If v' has degree 2 in G (Figure 9(c)), then by property (P) the edge $\{w', y\}$ is not redundant. In both cases, H contains all five edges $\{v, w\}, \{w, y\}, \{y, w'\}, \{w', v'\}$, and $\{v', v\}$. Deleting $\{v', v\}$ will therefore not disconnect the graph.

Case 3: Q is a pendant 3-ear.

Let $E(Q) = \{\{v, v'\}, \{v', w'\}, \{w', y\}\}$. Again, w has degree 2 in G. By (E5), there are the following two subcases.

Case 3.1: w' has degree 2 in G.

Then, again, by property (P) the edge $\{v, v'\}$ is not redundant, so H contains all five edges $\{v, w\}, \{w, y\}, \{y, w'\}, \{w', v'\}$, and $\{v', v\}$. Deleting $\{v', v\}$ will therefore not disconnect the graph (Figure 9(d)).

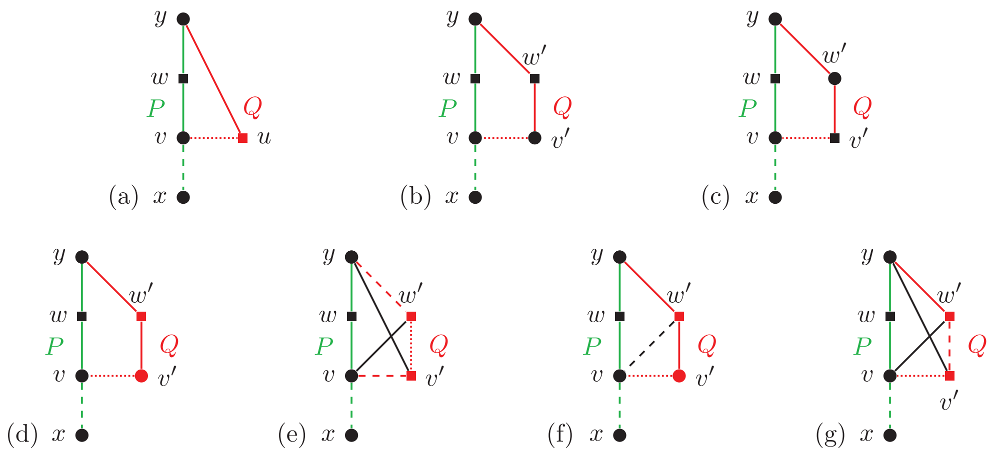

FIG. 9. *Proof of Lemma* 13: *A nonpendant* 3*-ear* P *(green) and the first nontrivial ear* Q *(red) attached to it. Dotted and solid edges belong to* H*; the dotted edge is deleted. Again, inner vertices of pendant ears are red, and vertices shown as squares have no incident edges other than those shown.*

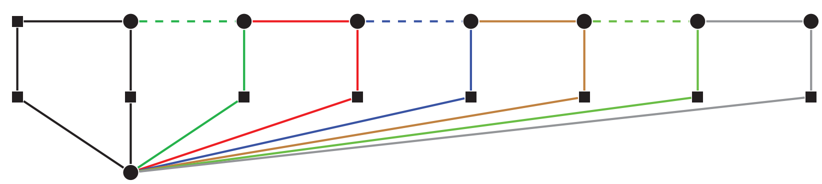

FIG. 10. *A* 2*-connected graph without redundant edges and an ear-decomposition with* (E1)–(E7)*, consisting of a closed* 5*-ear (left) and six* 3*-ears (constructed from left to right; the last one is pendant). The lower bound from Lemma* 13 *is* $n-1+5=21$*, but the* 20 *solid edges form a* 2*-edge-connected spanning subgraph.*

Case 3.2: $\Gamma(w') = \{y, v', v\}$ and $\Gamma(v') \subseteq \{v, w', y\}$.

As H was 2-connected, both v and y must be neighbors of $\{v', w'\}$ in H, and v' and w' must have degree at least 2 in H. Hence there are the following three subcases.

Case 3.2.1: H contains $\{v, w'\}, \{w', v'\}$, and $\{v', y\}$ (Figure 9(e)).

Case 3.2.2: H contains $\{v, v'\}, \{v', w'\}$, and $\{w', y\}$ (Figure 9(f)).

Case 3.2.3: H contains $\{v, w'\}, \{w', y\}, \{y, v'\}$, and $\{v', v\}$ (Figure 9(g)).

In each of the three subcases, deleting one of these edges will not disconnect the graph.

Indeed, in each case we deleted an edge from Q. Note that no edge from P or Q was deleted in any previous step, because these ears were never considered as Q before.

We conclude that after deleting k edges from H we still have a connected graph. Hence at least $n-1$ edges remain, so H had at least $n-1+k$ edges. $\square$

We remark that this is not a lower bound for 2EC, as the example in Figure 10 shows.

8. The approximation ratio. We will now show, assuming property (P), that the nontrivial ears in an ear-decomposition with (E1)–(E7) have at most $\frac{10}{7}\mathrm{OPT}(G)$ edges. In the proof we need to solve the following simple optimization problem.

LEMMA 14. *Let* $f : \mathbb{R}^7 \to \mathbb{R}$ *with*

$$f(a,b,c,d,e,n,\varphi) = \frac{\frac{5}{4}(n-1) + \frac{3}{4}\varphi + \frac{1}{2}(a+b+c+e)}{\max\{n-1+\varphi,\ 3a+4b+2c+2d+2e,\ n-1+b+c\}}.$$

Then

$$\max\bigl\{f(a,b,c,d,e,n,\varphi) : a,b,c,d,e,\varphi \geq 0,\ n \geq 2,\ 2a+3b+2c+5d+6e \leq n-1\bigr\} = \frac{10}{7}.$$

Proof. For $n = 15$, $a = 4$, $e = 1$, and $b = c = d = \varphi = 0$ we have $f(a,b,c,d,e,n,\varphi) = \frac{10}{7}$.

To show that the maximum is at most $\frac{10}{7}$, let $a,b,c,d,e,\varphi \geq 0$ and $n \geq 2$ with $2a+3b+2c+5d+6e \leq n-1$. We show $f(a,b,c,d,e,n,\varphi) \leq \frac{10}{7}$. Let

$$k := \max\{n-1+\varphi,\ 3a+4b+2c+2d+2e,\ n-1+b+c\}$$

be the denominator. Then $\frac{3}{4}(n-1+\varphi) \leq \frac{3}{4}k$ and $\frac{1}{2}(n-1+b+c) \leq \frac{1}{2}k$ and $\frac{1}{2}(a+e) = \frac{1}{28}((2a+6e)+4(3a+2e)) \leq \frac{1}{28}((2a+3b+2c+5d+6e)+4(3a+4b+2c+2d+2e)) \leq \frac{1}{28}((n-1)+4k) \leq \frac{5}{28}k$. Adding up these three inequalities yields

$$f(a,b,c,d,e,n,\varphi) \leq \frac{\frac{3}{4}k + \frac{1}{2}k + \frac{5}{28}k}{k} = \frac{10}{7}. \qquad \square$$

THEOREM 15. *There is a* $\frac{10}{7}$*-approximation algorithm for* 2*VC with running time* $O(n^3)$.

Proof. Let G be a given 2-connected graph. Due to Corollary 5 we can delete some redundant edges in $O(n^3)$ time so that the resulting spanning subgraph $\bar{G}$ of G satisfies property (P) and $\mathrm{OPT}(G) = \mathrm{OPT}(\bar{G})$. Then we use Corollary 10 to compute an ear-decomposition of $\bar{G}$ with (E1)–(E7) in $O(n^3)$ time. We delete the trivial ears and output the resulting 2-connected spanning subgraph H. We show that H has at most $\frac{10}{7}\mathrm{OPT}(G)$ edges.

Let a be the number of pendant 3-ears, and let b, c, d, and e denote the number of nonpendant 3-ears P such that the first nontrivial ear attached to P has length 2, 3, 4, and at least 5, respectively. (See Figure 3 again.) We claim that

$$(\star) \qquad |E(H)| \leq \frac{5}{4}(n-1) + \frac{3}{4}\varphi(\bar{G}) + \frac{1}{2}(a+b+c+e).$$

To show this, we sum over all ears, distinguishing cases as follows. For a 3-ear P whose first attached ear Q is a 4-ear, we have $|E(P)| + |E(Q)| = 7 = \frac{5}{4}(|\mathrm{in}(P)| + |\mathrm{in}(Q)|) + \frac{3}{4}$. Note that by (E4) no ear can be the first nontrivial attached ear of two 3-ears, so no 4-ear is counted twice here. For any other 3-ear P we have $|E(P)| = 3 = \frac{5}{4}|\mathrm{in}(P)| + \frac{1}{2}$. For any other even ear P we have $|E(P)| \leq \frac{5}{4}|\mathrm{in}(P)| + \frac{3}{4}$. Finally, for any odd ear P of length at least 5 we have $|E(P)| \leq \frac{5}{4}|\mathrm{in}(P)|$. Summation yields $(\star)$ because the numbers of inner vertices sum up to $n-1$ and by (E1) there are $\varphi(\bar{G})$ even ears.

Now we compare the upper bound in $(\star)$ to the lower bounds of the previous section. First, by Lemma 11, $\mathrm{OPT}(\bar{G}) \geq n - 1 + \varphi(\bar{G})$.

Let W_1 be the set of inner vertices of pendant 3-ears, W_2 the set of inner vertices of nonpendant 3-ears that have degree 2, and W_3 the set of inner vertices of 2-ears that are the first nontrivial ear attached to a 3-ear. Let $W := W_1 \cup W_2 \cup W_3$. By

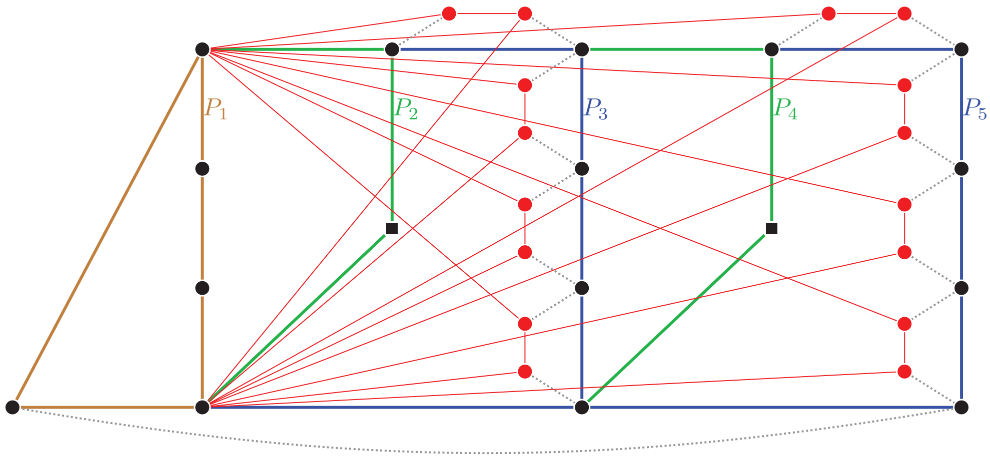

FIG. 11. *A tight example. For each $k \in \mathbb{N}$ there is a graph G_k with $14k + 5$ vertices and property* (P) *and an ear-decomposition of G_k as follows (we show $k = 2$). There are a closed 5-ear P_1 (brown), for $i = 1, \ldots, k$ a 3-ear P_{2i} (green), and a 5-ear P_{2i+1} (blue), and then $4k$ pendant 3-ears (red) and $8k+1$ trivial ears (dotted) such that* (E1)–(E7) *are satisfied. Deleting the trivial ears yields a minimal 2-connected spanning subgraph with $20k+5$ edges. However, G_k has a Hamiltonian circuit (using all the dotted edges), so $\mathrm{OPT}(G_k) = |V(G_k)| = 14k + 5$.*

(E6) we have $|W| = 2a + 2b + c + d + e$. Then consider $\bar{G}[W]$, the subgraph of $\bar{G}$ induced by W. By (E3) and (E7), this subgraph has no edges between inner vertices of different ears. Hence it has $q_W = a + 2b + c + d + e$ connected components. By Lemma 12 we have $\mathrm{OPT}(\bar{G}) \geq |W| + q_W = 3a + 4b + 2c + 2d + 2e$.

Finally, by Lemma 13, we have $\mathrm{OPT}(\bar{G}) \geq n - 1 + b + c$. Combining the three inequalities, we conclude

$$(\star\star) \qquad \mathrm{OPT}(\bar{G}) \geq \max\{n - 1 + \varphi(\bar{G}),\, 3a + 4b + 2c + 2d + 2e,\, n - 1 + b + c\}.$$

Moreover, $2a + 3b + 2c + 5d + 6e \leq n - 1$ by summing over the 3-ears, taking each together with the first nontrivial ear attached to it unless it is a 3-ear itself (again, by (E4), no nontrivial ear can be the first attached ear of two 3-ears).

From $(\star)$, Lemma 14, and $(\star\star)$ we get

$$\begin{aligned} |E(H)| &\leq \frac{5}{4}(n-1) + \frac{3}{4}\varphi(\bar{G}) + \frac{1}{2}(a + b + c + e) \\ &\leq \frac{10}{7}\max\{n - 1 + \varphi(\bar{G}),\, 3a + 4b + 2c + 2d + 2e,\, n - 1 + b + c\} \\ &\leq \frac{10}{7}\mathrm{OPT}(\bar{G}) \\ &= \frac{10}{7}\mathrm{OPT}(G). \end{aligned}$$

□

Our analysis is tight; see Figure 11. We also remark that for graphs with minimum degree 3 we immediately get the better approximation ratio $\frac{17}{12}$ (as in [1]) because (E6) then implies $b = c = d = e = 0$, and $\frac{\frac{5}{4}(n-1)+\frac{3}{4}\varphi+\frac{1}{2}a}{\max\{n-1+\varphi,\, 3a,\, n-1\}} \leq \frac{17}{12}$ for all $n, \varphi, a \geq 0$.

Appendix: Counterexample to the Vempala–Vetta approach. We will now show that the approach of Vempala and Vetta [13], who claimed to have a $\frac{4}{3}$-approximation algorithm, does not work without modifications. As lower bound they

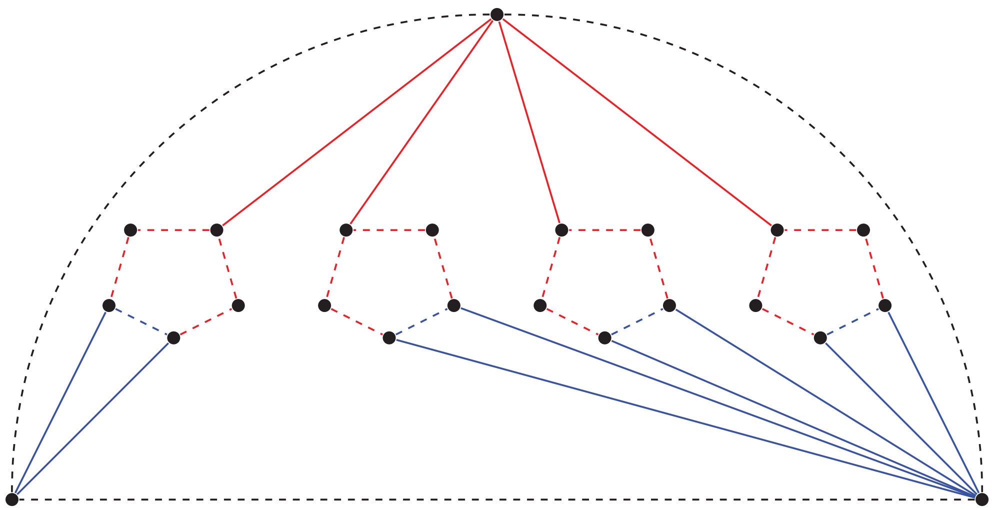

FIG. 12. *Counterexample to* [13]. *For every $k \in \mathbb{N}$ there is a 2-connected graph G_k with $5k+3$ vertices and the following properties (we show $k=4$). There is no beta, and there are no adjacent degree-2 vertices. The dashed edges form a 2-regular subgraph, so $L_{D2} = n = 5k+3$. Every 2-connected spanning subgraph must contain all red edges and two out of the three blue edges incident to each dashed pentagon. So $\mathrm{OPT}(G_k) \geq 7k$. For $k > 12$ the ratio is worse than $\frac{4}{3}$.*

use the minimum number L_{D2} of edges of a spanning subgraph in which every vertex has degree at least 2. Moreover, in a 2-connected graph G, they define a *beta* (vertex or pair) to be a set $C \subseteq V(G)$ with $|C| \leq 2$ that is one of at least three connected components of $G - \{s,t\}$ for some $s,t \in V(G)$.

If G is a 2-connected graph without adjacent degree-2 vertices and without beta, they claim that their algorithm finds a 2-connected spanning subgraph with at most $\frac{4}{3}L_{D2}$ edges. However, such a subgraph does not always exist; see Figure 12.

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