Submodular Function Minimization under Congruency Constraints

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- Submodular function: Set function $f\colon 2^N \to \mathbb{R}$ on finite set N such that

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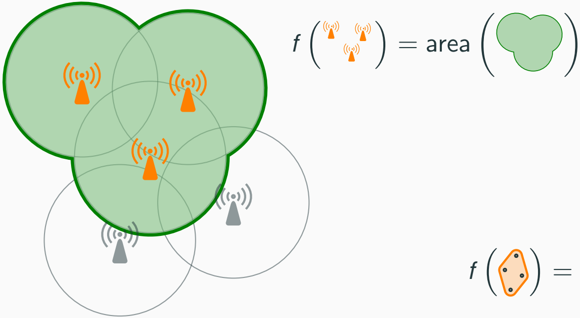
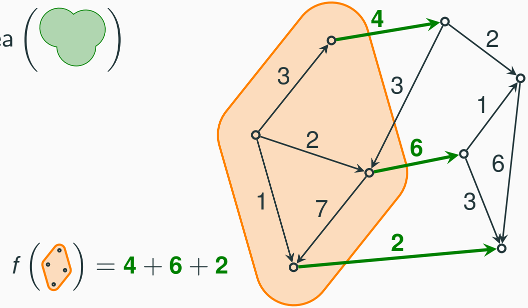
- Examples: coverage functions, cut functions, rank functions for matroids.





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- History of SFM algorithms:
  - *1981:* Weakly polynomial (ellipsoid-based) [Grötschel, Lovász, Schrijver].
  - *1985:* Combinatorial pseudo-polynomial [Cunningham].
  - *1988:* Strongly polynomial (ellipsoid-based) [Grötschel, Lovász, Schrijver].
  - *1999:* Combinatorial strongly polynomial [Iwata, Fleischer, Fujishige] and [Schrijver].
  - *2009 – :* Speedups [Orlin 2009], [Lee, Sidford, Wong 2015], [Chakraborty, Lee, Sidford, Wong 2017], [Dadush, Végh, Zambelli 2018].

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Under what constraints is efficient submodular function minimization possible?

Constrained submodular minimization: Prior results

- Parity-constrained SFM:

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- Captured by more general constraint families over which SFM can be done efficiently:
  - *Triple families* [Grötschel, Lovasz, Schrijver 1984].
  - *Parity families* [Goemans, Ramakrishnan 1995].

- Long-standing open problem:

Can p-modular ILPs be solved efficiently?

  - Well-known for unimodular systems.
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- Open questions [Geelen, Kapaida 2017]:

t-Set Even-Cut Problem	t-Set Odd-Cut Problem
Let $G = (V, E)$ a graph and $T_1, \ldots, T_t \subseteq V$. Find a non-empty $S \subsetneq V$ s.t. $\lvert S \cap T_i \rvert$ are all even and $\lvert\delta(S)\rvert$ is minimized.	Let $G = (V, E)$ a graph and $T_1, \ldots, T_t \subseteq V$. Find a non-empty $S \subsetneq V$ s.t. $\lvert S \cap T_i \rvert$ are all odd and $\lvert\delta(S)\rvert$ is minimized.

- **Congruency-Constrained Submodular Minimization (CCSM):**

  Let $f\colon 2^N \to \mathbb{Z}$ be submodular, let $m \in \mathbb{Z}_{>0}$, and let $r \in \{0, \ldots, m-1\}$.

$$\begin{aligned} \min\ & f(S) \\ \text{s.t.}\quad & S \subseteq N, \\ & |S| \equiv r \pmod{m}. \end{aligned} \tag{CCSM}$$

Theorem 1: Solving CCSM

For any $m \in \mathbb{Z}_{>0}$ that is a prime power, (CCSM) can be solved in time $|N|^{2m+\mathcal{O}(1)}$.

- **Generalised Congruency-Constrained Submodular Minimization (GCCSM):**

  Let $f\colon 2^N \to \mathbb{Z}$ submodular, $m \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_k \in \{0, \ldots, m-1\}$, and $S_1, \ldots, S_k \subseteq N$.

$$\begin{aligned} \min\ & f(S) \\ \text{s.t.}\quad & S \subseteq N, \\ & |S \cap S_i| \equiv r_i \pmod{m} \quad \forall i \in [k]. \end{aligned} \tag{GCCSM}$$

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- Captures both the *t*-Set Even-Cut Problem and the *t*-Set Odd-Cut Problem.

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$\mathrm{Enum}(d)$: Enumeration algorithm of depth d for CCSM

1. For all disjoint $A, B \subseteq N$ with $|A|, |B| \leqslant d$, find a minimal minimizer of f over

$$\mathcal{L}_{AB} := \{S \subseteq N \mid A \subseteq S \subseteq N \setminus B\} \ .$$

   Let $\mathcal{S}$ contain one minimal minimizer for each pair (A, B).

2. Among all $S \in \mathcal{S}$ with $|S| \equiv r \pmod{m}$, return best one.

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- $\mathrm{Enum}(d)$ is extension of algorithm in [Goemans, Ramakrishnan 1995].

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Definition: (m, d)-system

For a finite set N, a family $\mathcal{H} \subseteq 2^N$ is called (m, d)-system on N if

(i) $\mathcal{H}$ is closed under intersection,

(ii) $|H| \not\equiv |N| \pmod{m}$ $\forall H \in \mathcal{H}$, and

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Theorem 4: Inexistence of systems

For $m \in \mathbb{Z}_{>0}$ being a prime power, there is no $(m, m-1)$-system.

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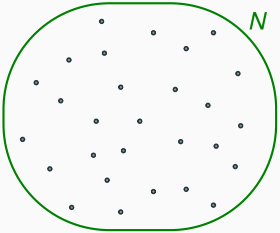

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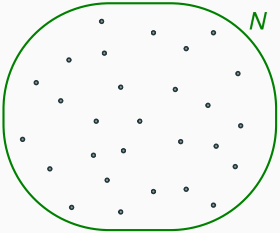

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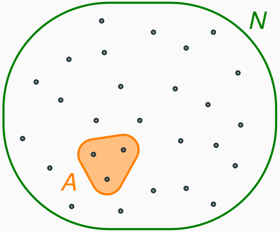

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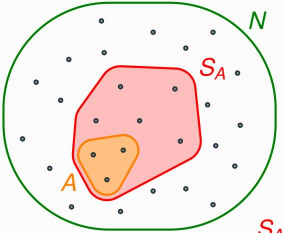

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Claim

The following family $\mathcal{H}$ is an (m, d)-system:

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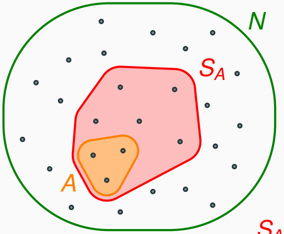

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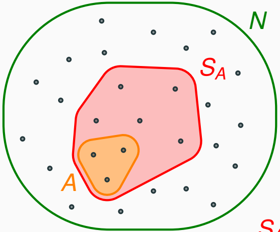

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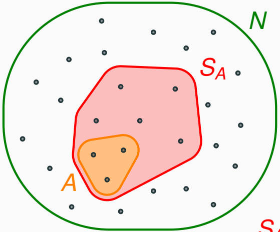

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ad (ii):

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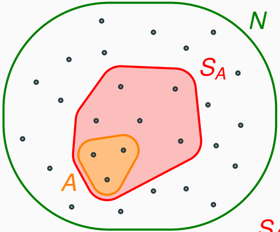

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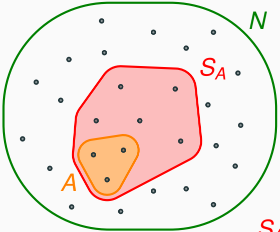

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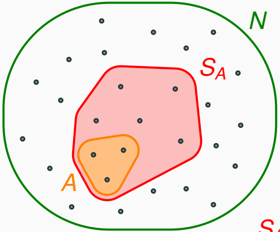

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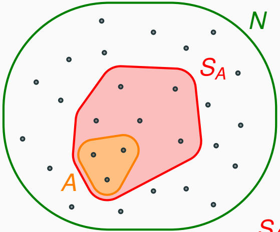

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First steps: No $(2, 1)$-system exists

Properties of a $(2, 1)$-system $\mathcal{H}$

- Closed under intersections.
- $|H| \not\equiv |N| \pmod 2$ for all $H \in \mathcal{H}$.
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Step 2: Contradiction by inclusion-exclusion principle:

$$|N| \underset{\text{covering property}}{=} \left| \bigcup_{H \in \mathcal{H}} H \right| \overset{\text{inclusion-exclusion}}{=} \sum_{\ell=1}^{|\mathcal{H}|} (-1)^{\ell+1} \sum_{\substack{H_1, \ldots, H_\ell \in \mathcal{H}, \\ \forall i \neq j:\ H_i \neq H_j}} \left| \bigcap_{i=1}^{\ell} H_i \right| \underset{\text{Step 1}}{\equiv} 0 \pmod 2 .$$

Proof plan to show that no $(p, p-1)$-system exists

Properties of an (m, d)-system $\mathcal{H}$

- Closed under intersections.
- $|H| \not\equiv |N| \pmod{m}$ for all $H \in \mathcal{H}$.
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If there exists a $(p, p-1)$-system, then there exists a $(p, 1)$-system such that

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- Exploit inclusion-exclusion again for contradiction:

$$|N| = \left| \bigcup_{H \in \mathcal{H}} H \right| = \sum_{\ell=1}^{|\mathcal{H}|} (-1)^{\ell+1} \sum_{\substack{H_1, \ldots, H_\ell \in \mathcal{H}, \\ \forall i \neq j:\ H_i \neq H_j}} \left| \bigcap_{i=1}^{\ell} H_i \right| \equiv 0 \pmod{p} \ .$$

Step 1: Assume $(p, p-1)$-system $\mathcal{H}$ with $|N| \equiv 0 \pmod{p}$.

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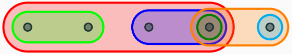

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Step 2: Transform sets to $(p-1)$-fold cartesian product

- Ground set cardinality: $|N|^{p-1} \equiv 0 \pmod{p}$.
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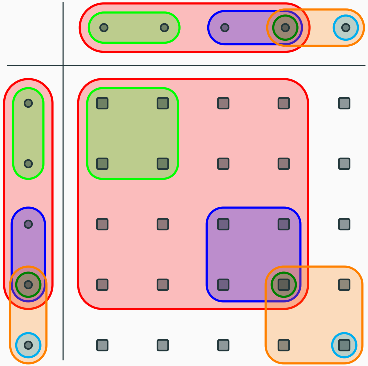

Example: 2-fold product for $p = 3$

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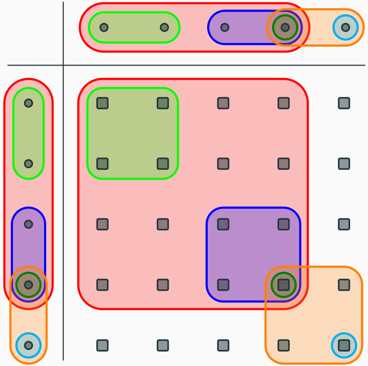
Step 3: Shift to obtain $|H| \equiv 0 \pmod p$ for all $H \in \mathcal{H}$.

- By adding $p-1$ elements.



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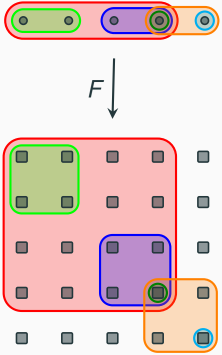

- Key ingredient: Set system transformation function F.
- Crucial properties:
  - Cardinality transformation:
  $$|F(S)| \equiv \begin{cases} 0 \pmod{m} & \text{if } |S| \equiv 0 \pmod{m}, \\ 1 \pmod{m} & \text{if } |S| \not\equiv 0 \pmod{m}. \end{cases}$$
  - Preserving intersections:
  $$F(S) \cap F(T) = F(S \cap T).$$
  - Preserving coverage.
- Feasible functions:

  polynomial: $|F(S)| = |S|^k$, *binomial:* $|F(S)| = \binom{|S|}{k}$, *conic combinations thereof*
- Function used for moduli m that are prime powers:
$$|F(S)| = \sum_{\substack{1 \leq k < m, \\ k \text{ odd}}} \binom{|S|}{k} + (p-1) \cdot \sum_{\substack{1 \leq k < m, \\ k \text{ even}}} \binom{|S|}{k} .$$

- Main results: Polynomial-time algorithms for

CCSM

$$\begin{aligned} &\min f(S) \\ \text{s.t.} \quad & S \subseteq N, \\ & |S| \equiv r \pmod{m}. \end{aligned}$$

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- Barrier: $(m, m-1)$-systems do exist for composite m [Gopi 2017].