Advances on Strictly Δ-Modular IPs

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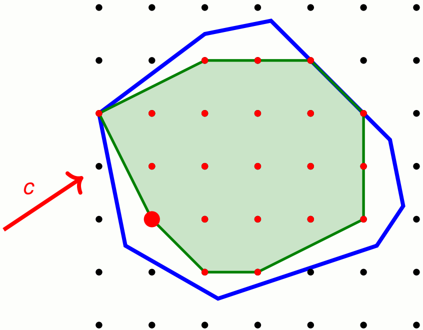
Integer Programming



Integer Linear Programming (IP)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve

$$\min\{c^\top x : Ax \leqslant b,\ x \in \mathbb{Z}^n\} \ .$$

An interesting class of efficiently solvable IPs

A totally unimodular (TU) $\implies$ Integral relaxation.

What if minors, in absolute value, are still bounded, but not by 1?

Bounded subdeterminants

Δ-modular Integer Programming

Can IPs with Δ-modular constraint matrix be solved efficiently for constant $\Delta \in \mathbb{Z}_{>0}$?

- $A \in \mathbb{Z}^{m \times n}$ is Δ-modular if
  - $\rightarrow$ $\text{rank}(A) = n$
  - $\rightarrow$ $n \times n$ subdets bounded by Δ
- less general:
  - $\rightarrow$ total Δ-modularity: bounds on *all* subdets
  - $\rightarrow$ strict Δ-modularity: subdets in $\{0, \pm\Delta\}$ only

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Poly-time solvable special cases

✓ $\Delta = 1$: Immediate

✓ $\Delta = 2$: Bimodular Integer Programming (BIP)
[Artmann, Weismantel, and Zenklusen, STOC 2017]

✓ Totally Δ-modular IPs, at most 2 non-zeros per row
[Fiorini, Joret, Weltge, and Yuditsky, FOCS 2021]

✓ Feasibility for strictly 3-modular IPs (randomized)
[Nägele, Santiago, and Zenklusen, SODA 2022]

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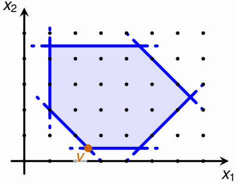
Our main result

Strongly polynomial randomized alg. for feasibility of strictly 4-modular IPs.

High-level view: BIP and strictly 3-modular feasibility



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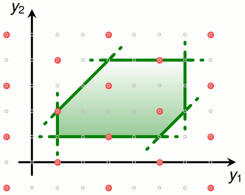

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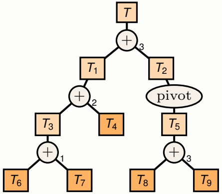
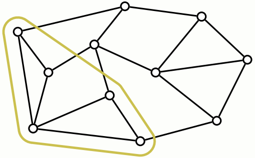


Seymour's TU decomposition

Reduction to base block problems.



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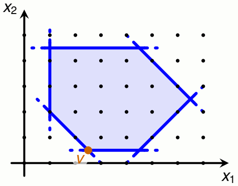

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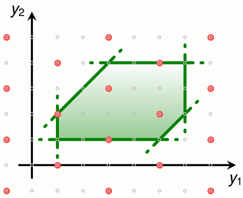

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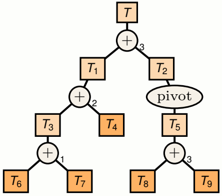

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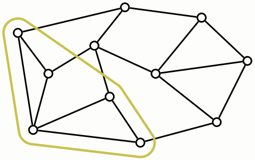

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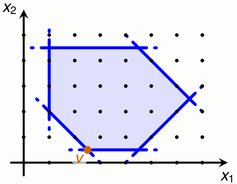

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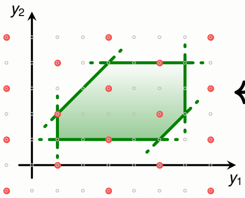

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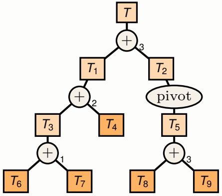

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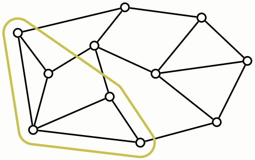

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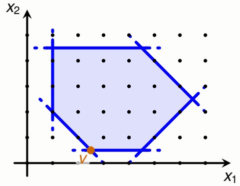

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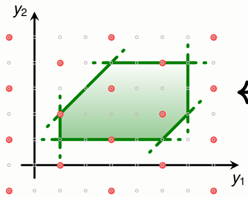

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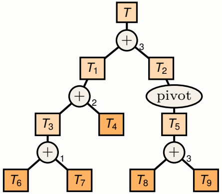

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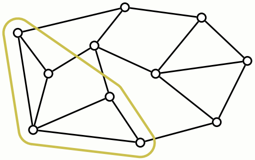

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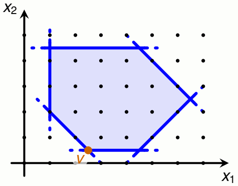

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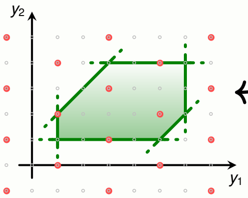

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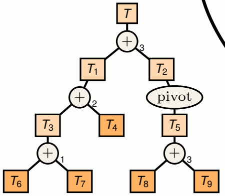

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Cut baseblock:

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Circulation baseblock:

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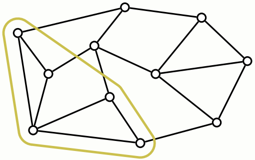

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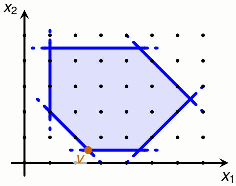

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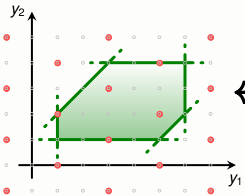

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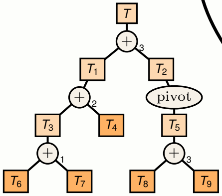

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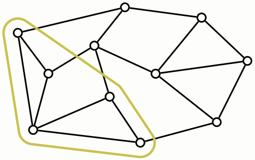

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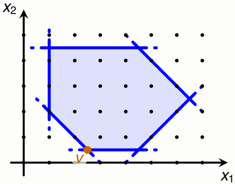

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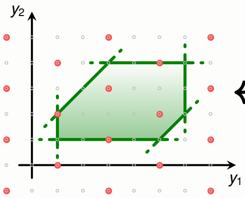

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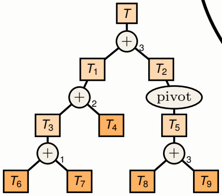

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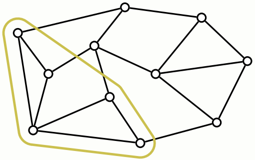

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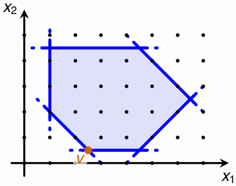

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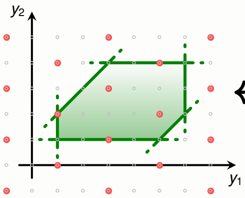

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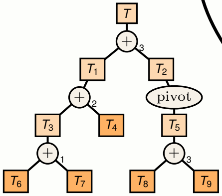

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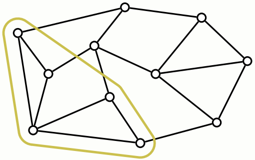

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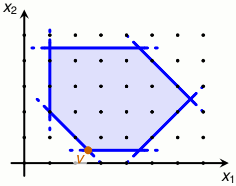

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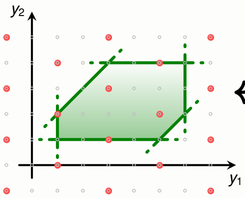

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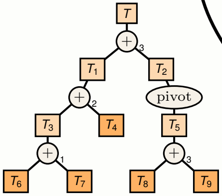

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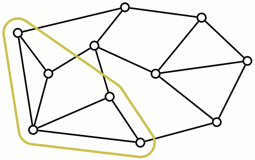

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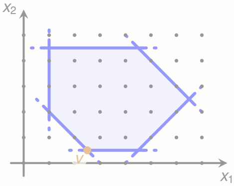

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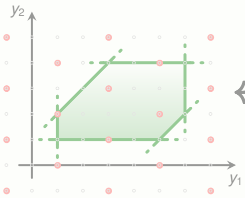

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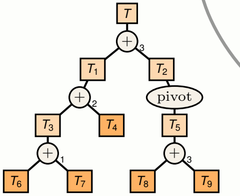

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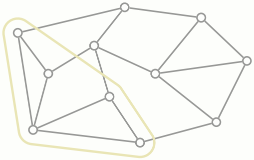

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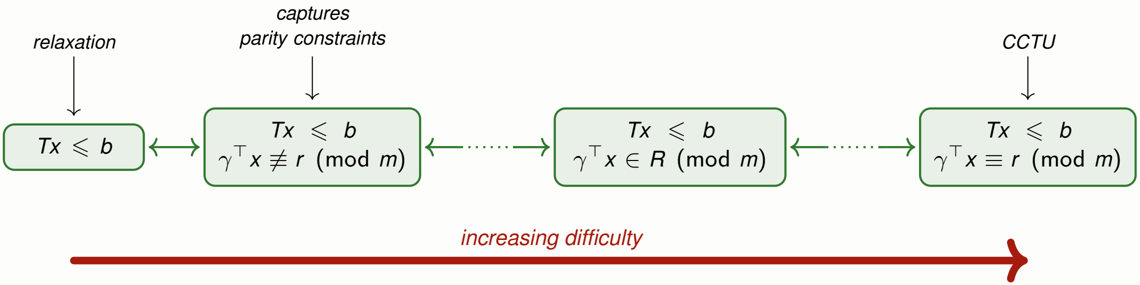
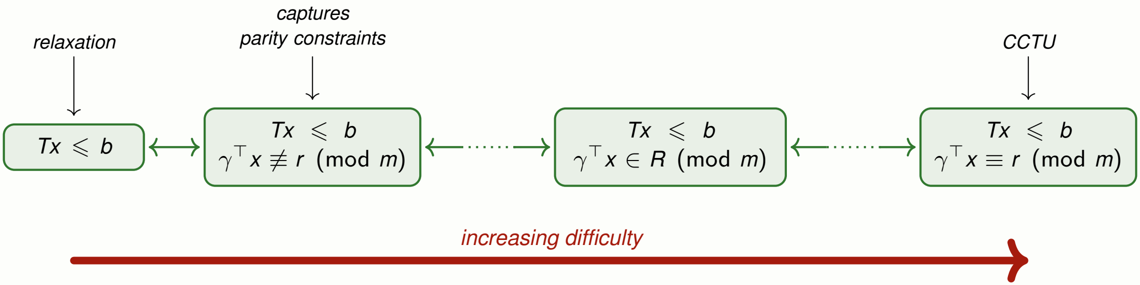
A hierarchy of congruency-constrained TU problems



A hierarchy of congruency-constrained TU problems



Old results

- ✓ Optimization for depth one
- ✓ Feasibility for depth two if m is prime

New result

- ✓ Feasibility for depth three and general m

Exploiting the hierarchy with Cauchy-Davenport

$$\begin{pmatrix} A & \textit{rk 1} \\ \textit{rk 1} & B \end{pmatrix} x \leqslant b$$

$$\gamma^\top x \in \boxed{0}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}\,\boxed{6} \pmod 7$$

depth 3

Exploiting the hierarchy with Cauchy-Davenport

$$\begin{pmatrix} A & \text{rk } 1 \\ \text{rk } 1 & B \end{pmatrix} x \leqslant b$$

$\gamma^\top x \in$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\pmod 7$

depth 3

A-subproblem
$Ax_A \leqslant b_A$

B-subproblem
$Bx_B \leqslant b_B$

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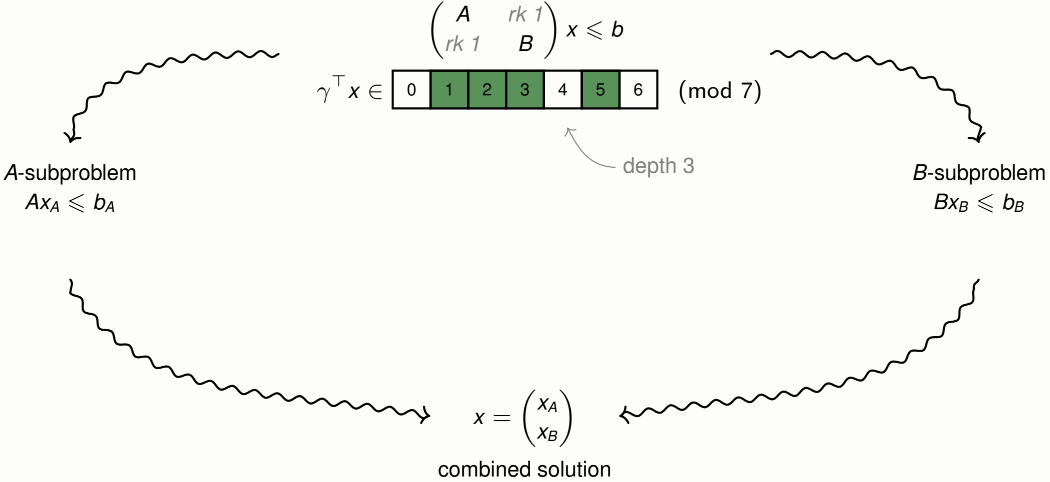

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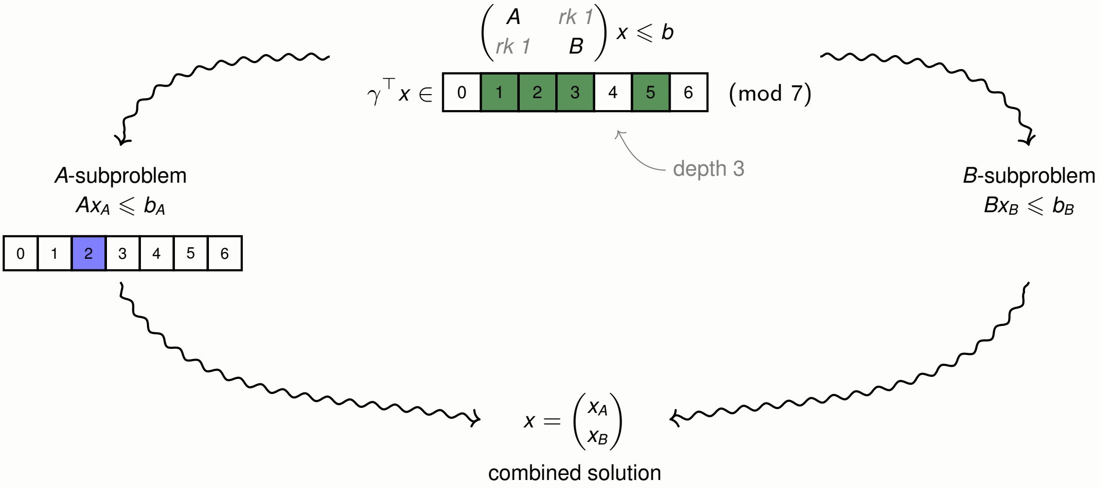

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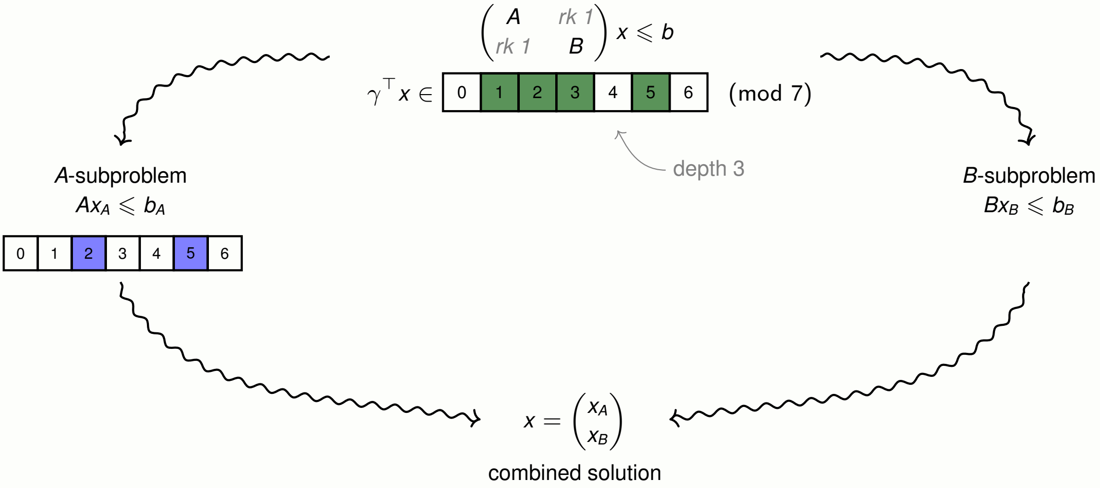

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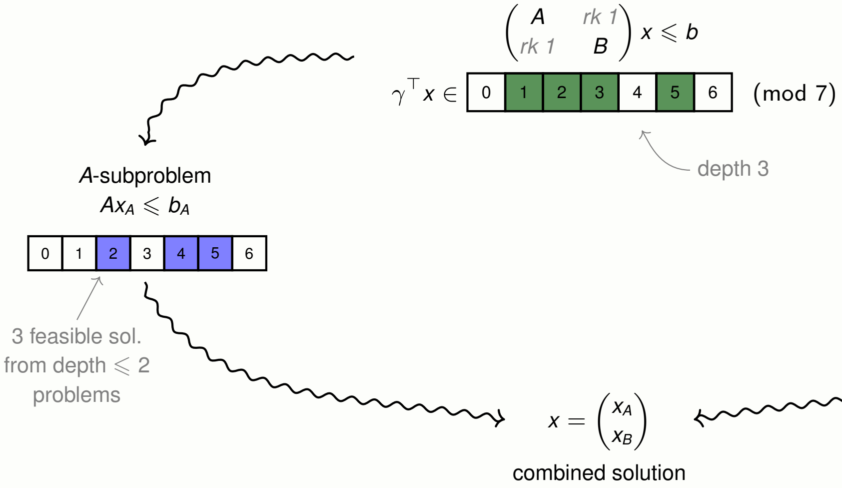

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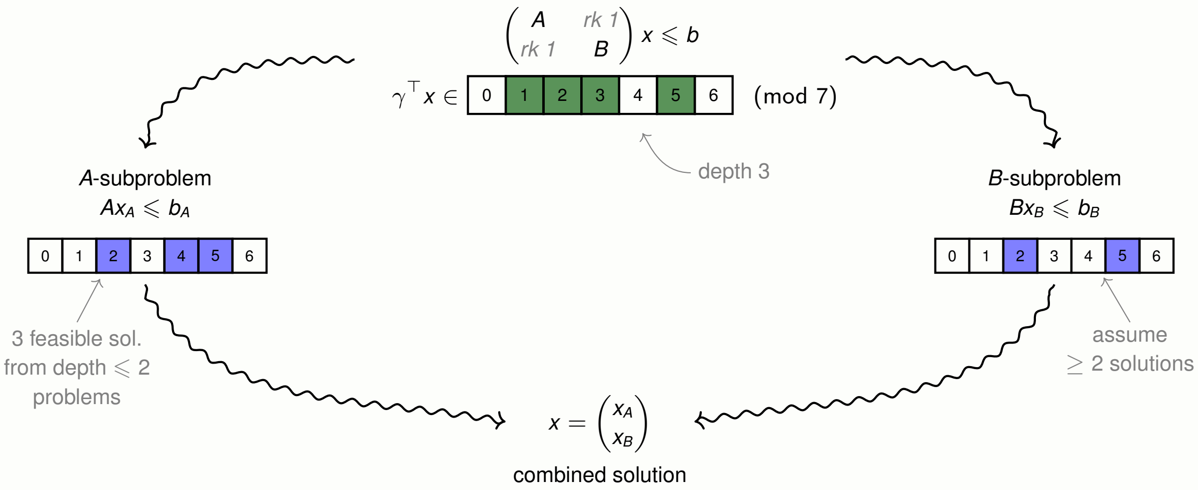

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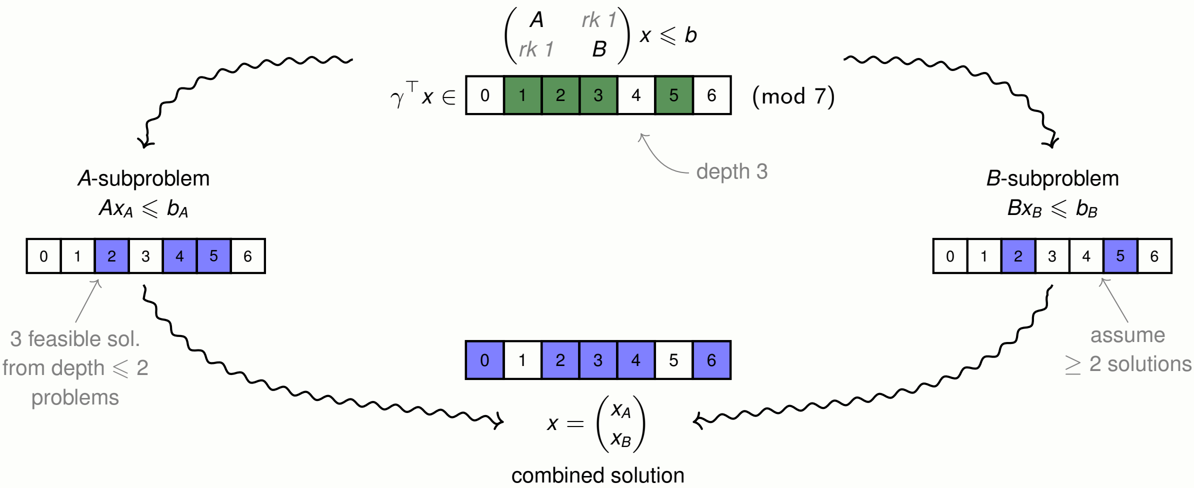

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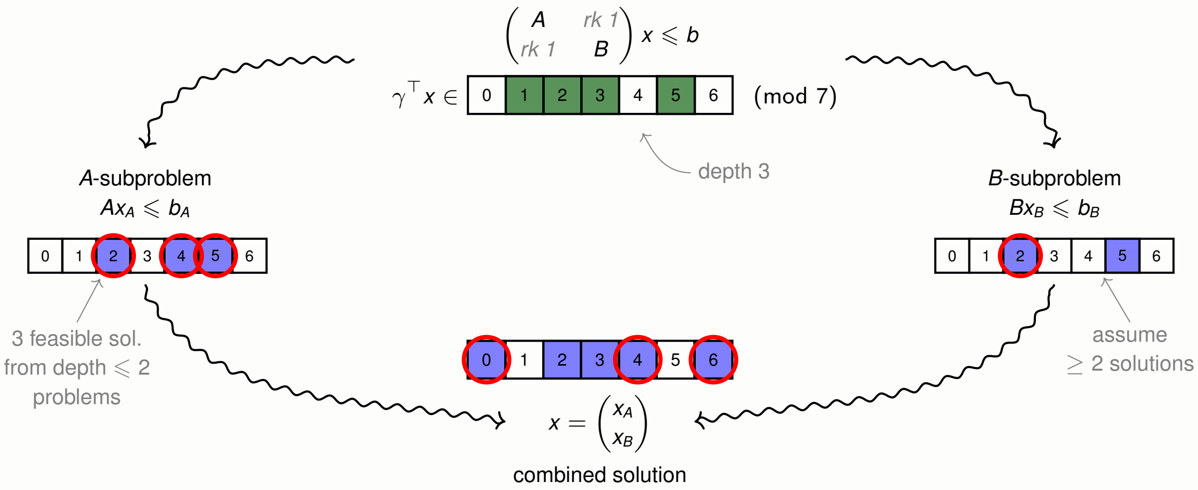

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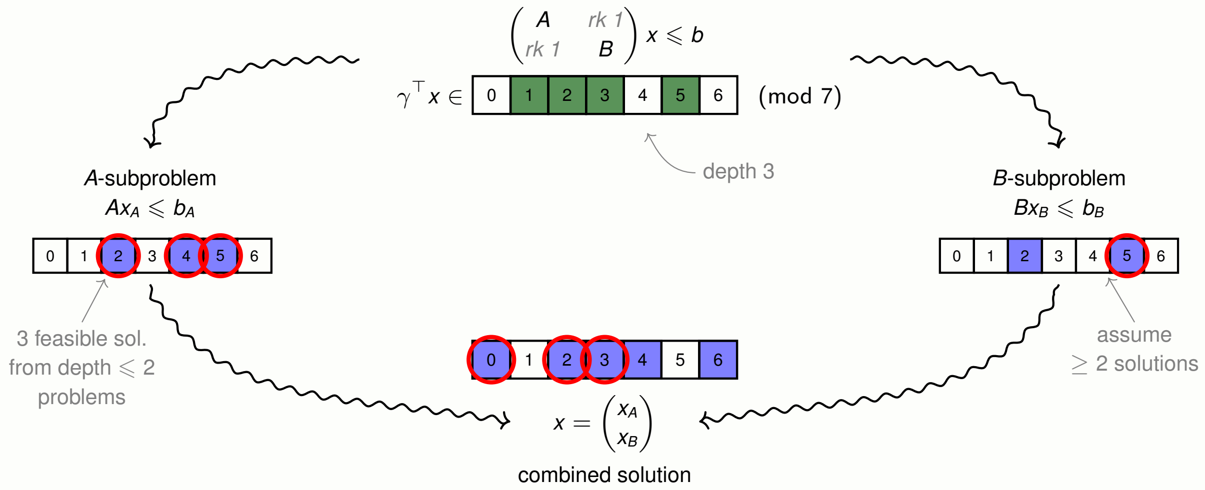
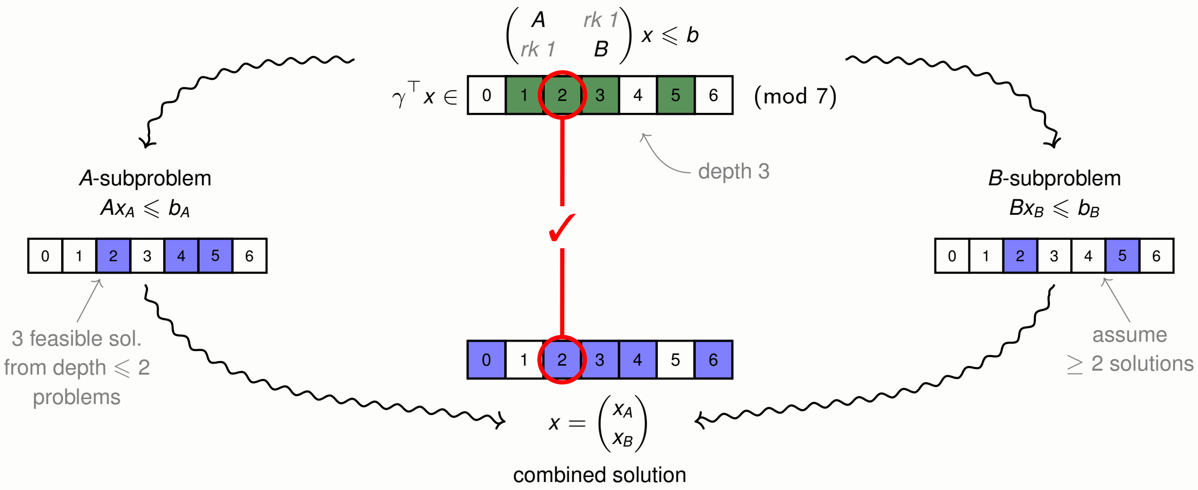
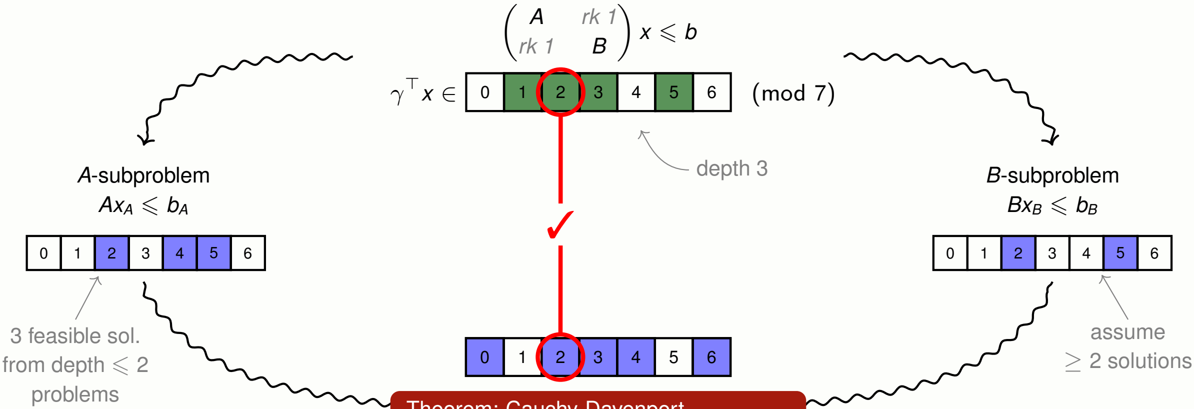


Exploiting the hierarchy with Cauchy-Davenport



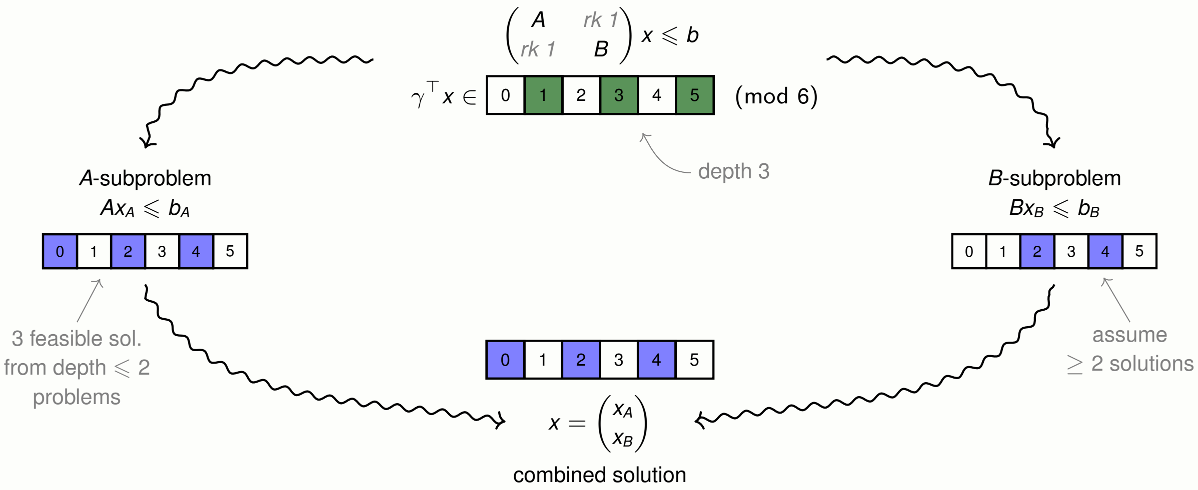
Exploiting the hierarchy with Cauchy-Davenport



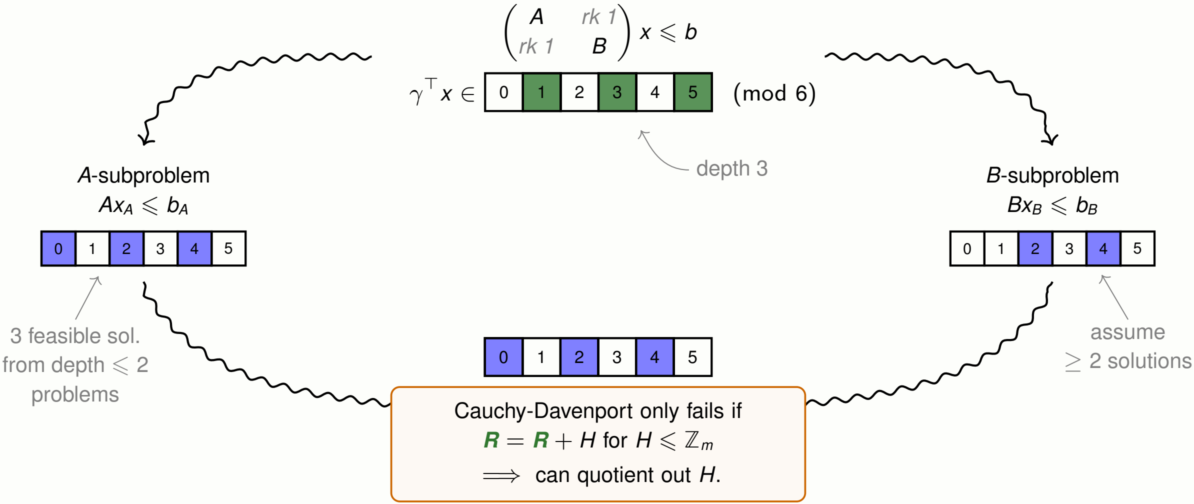
Theorem: Cauchy-Davenport

For prime m and $R_A, R_B \subseteq \mathbb{Z}_m$,
$|R_A + R_B| \geq \min\{m, |R_A| + |R_B| - 1\}$.

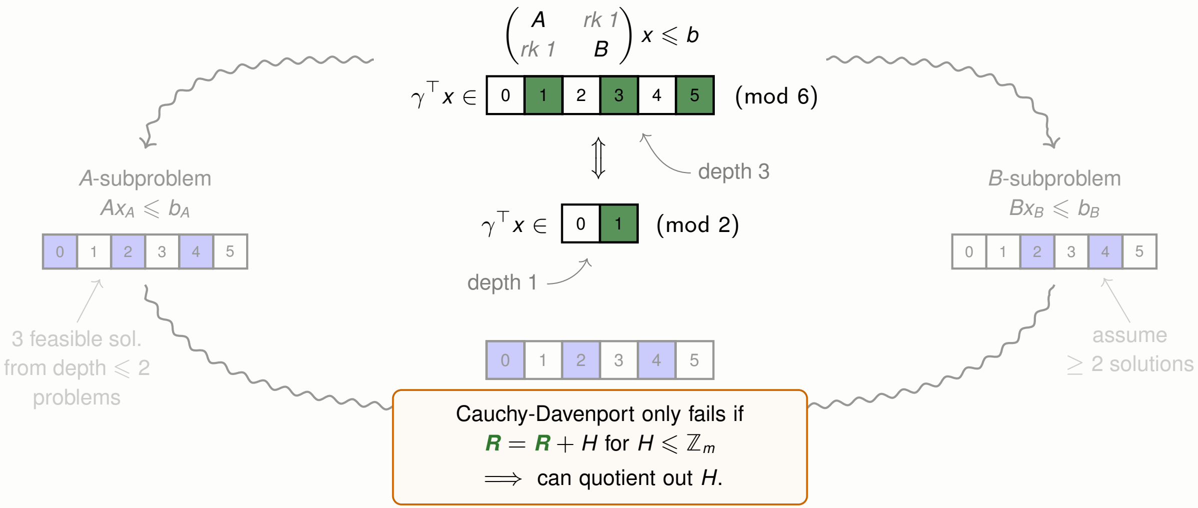
Non-prime modulus



Non-prime modulus



Non-prime modulus



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arXiv preprint coming soon!