A 1.5-Approximation for Path TSP

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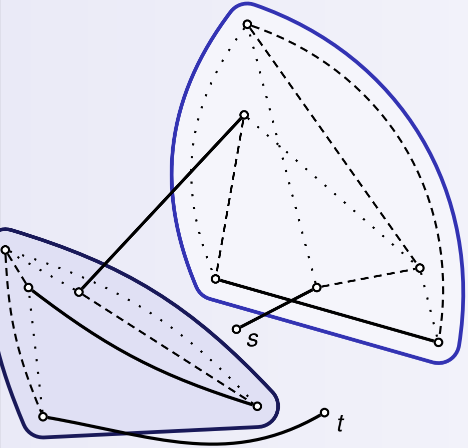
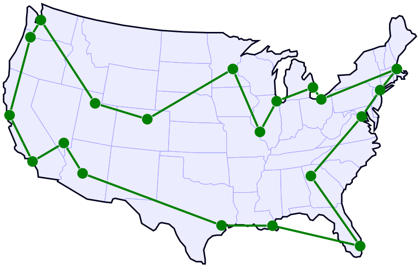
Presentation: Martin Nägele, ETH Zurich

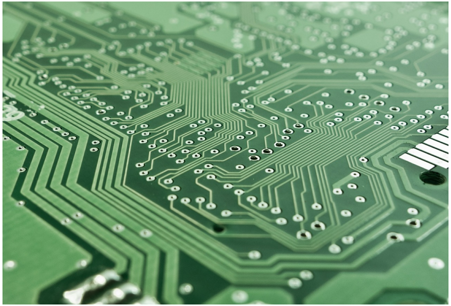


A brief intro to the Traveling Salesman Problem

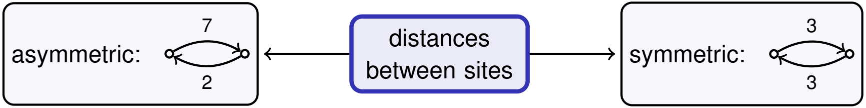
The (Metric) Traveling Salesman Problem (TSP)

What's the quickest way to visit n sites?

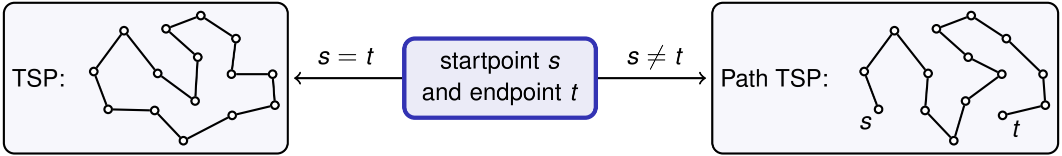




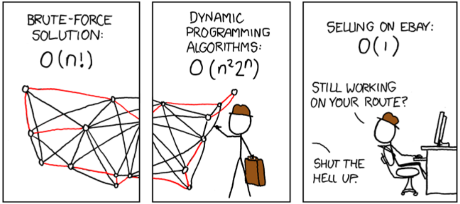
Common variations of TSP



- Complete graph $G = (V, E)$.
- Metric length $\ell \colon E \to \mathbb{R}_{\geqslant 0}$.



- All variants are well-known to be APX-hard.



(xkcd.com / CC BY-NC 2.5)

- Major open problem what efficient computation can achieve.

TSP	Path TSP
	1.667 [Hoogeveen, 1991]
	1.618 [An, Kleinberg, Shmoys, 2012]
	1.6 [Sebő, 2013]
1.5 [Christofides, 1978]	1.599 [Vygen, 2016]
	1.566 [Gottschalk, Vygen, 2016]
	1.529 [Sebő, van Zuylen, 2016]
	$1.5 + \varepsilon$ [Traub, Vygen, 2018a]

Exciting progress for graph metrics:

[Oveis Gharan, Saberi, Singh, 2011]

[Mucha, 2014]

[Sebő, Vygen, 2014]

[Mömke, Svensson, 2016]

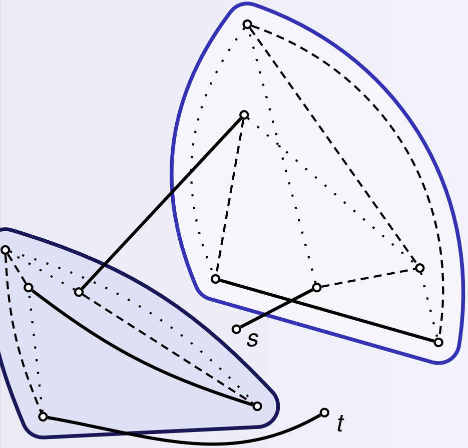
[Traub, Vygen, 2018b]

[. . .]

There is a 1.5-approximation for Path TSP.

- We move away from prior approaches, which focussed on so-called *narrow cuts*.
- Technical ingredients: Obtain a strong Held-Karp solution z using
  - Karger's bound on the number of near-min cuts, and
  - Dynamic programming "à la Traub & Vygen".

  Run a Christofides-type algorithm with a spanning tree obtained from z.
- Analysis follows Wolsey's approach.
- Natural barrier 1.5: Any progress improves upon Christofides' 1.5-approximation for TSP.



Following in Christofides' footsteps

Why it works for TSP but fails for Path TSP. . .

(Spoiler: . . . and can be fixed.)

- Find connected Eulerian graph with good total length, exploit metric lengths to shortcut.

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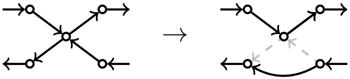

The general idea

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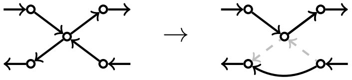


Christofides' 1.5-approximation for TSP

1. Find a shortest spanning tree T.

$$\implies \ell(T) \leqslant \ell(\mathrm{OPT}) \ .$$

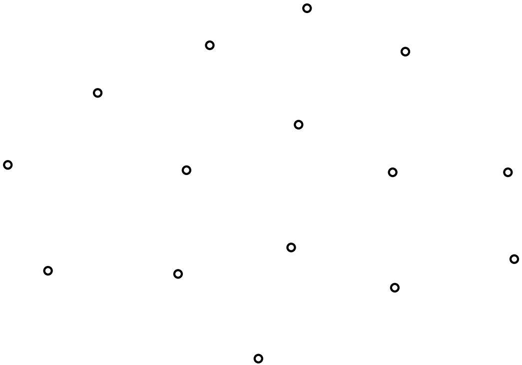
2. Find a shortest odd(T)-join J.

$$\implies \ell(J) \leqslant \frac{1}{2} \cdot \ell(\mathrm{OPT}) \ .$$

3. Find Eulerian tour in multiunion of T and J.

4. Return shortcutted Hamiltonian tour H.

$$\implies \ell(H) \leqslant \ell(T) + \ell(J) \leqslant \frac{3}{2} \cdot \ell(\mathrm{OPT}) \ .$$



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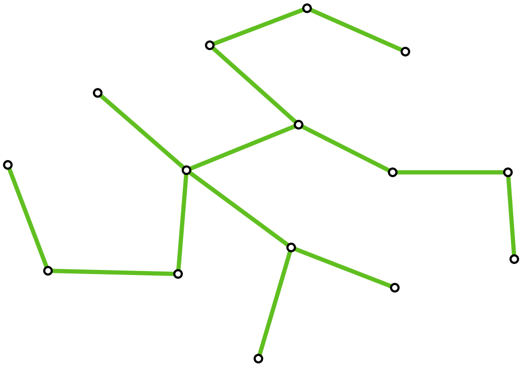

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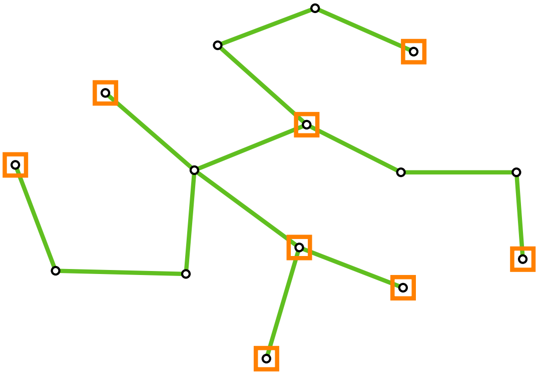

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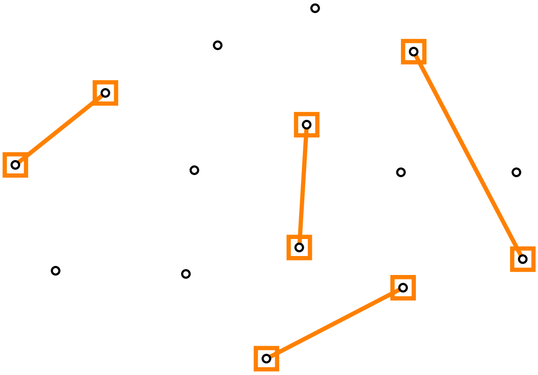

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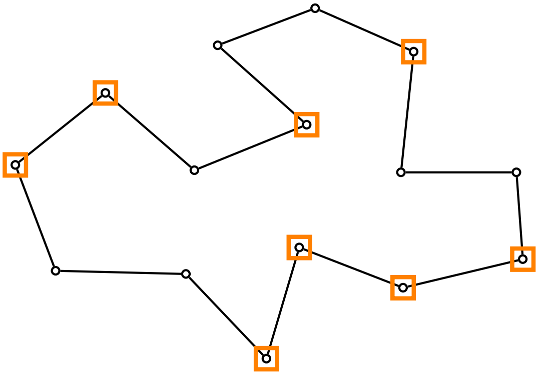

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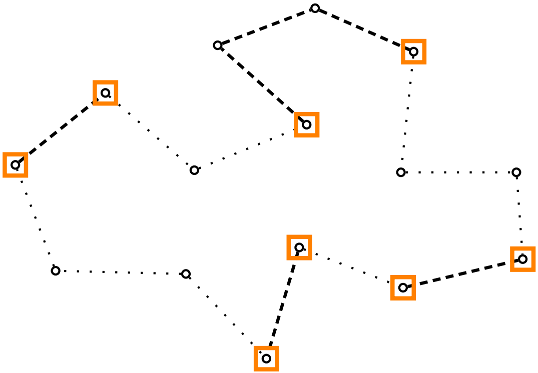

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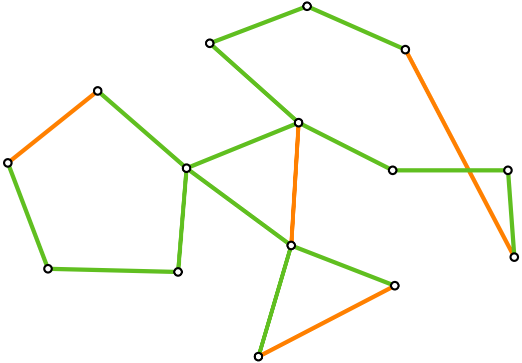

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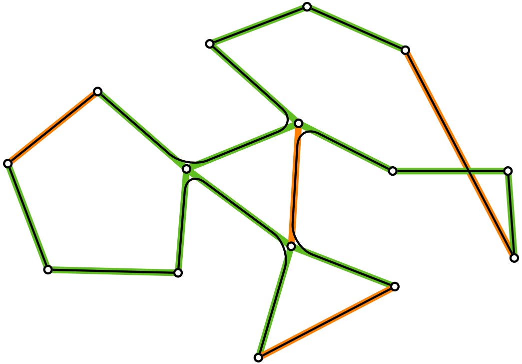

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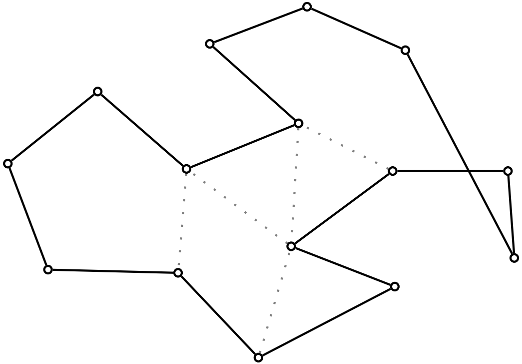
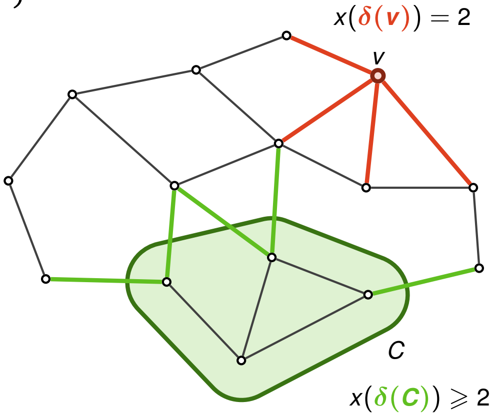


- Held-Karp polytope

$$P_{\mathrm{HK}} := \left\{ x \in \mathbb{R}^{E}_{\geqslant 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 2 & \forall v \in V \\ x(\delta(C)) \geqslant 2 & \forall C \subsetneq V,\ C \neq \emptyset \end{array} \right\} .$$

- Held-Karp relaxation

$$\min\{\ell^{\top} x \mid x \in P_{\mathrm{HK}}\} .$$



- Let $x^* \in \operatorname{argmin}\{\ell^\top x \mid x \in P_{\mathrm{HK}}\}$.

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Claim

If T is a shortest spanning tree, and J is a shortest odd(T)-join, then

(a) $\ell(T) \leqslant \ell^\top x^*$, and ***(b)*** $\ell(J) \leqslant \frac{1}{2} \cdot \ell^\top x^*$.

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- Shows 1.5-approximation and upper bound on integrality gap.

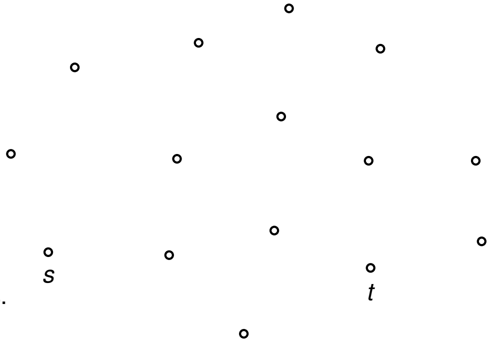
Christofides' approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $\boldsymbol{T}$: $\ell(\boldsymbol{T}) \leqslant \ell(\mathrm{OPT})$.
- But: OPT does not contain two disjoint Q_T-joins.
- Still, shortest Q_T-join $\boldsymbol{J}$ satisfies

$$\ell(\boldsymbol{J}) \leqslant \tfrac{2}{3} \cdot \ell(\mathrm{OPT}). \qquad \text{[Hoogeveen, 1991]}$$

  Proof: Together, OPT and T contain three Q_T-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.



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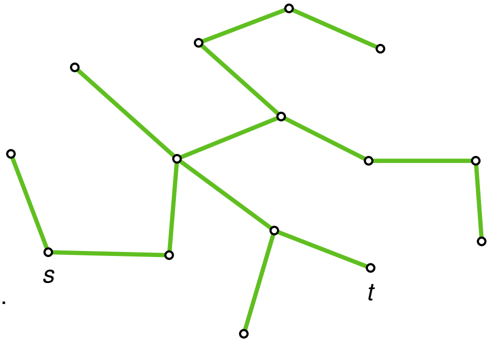

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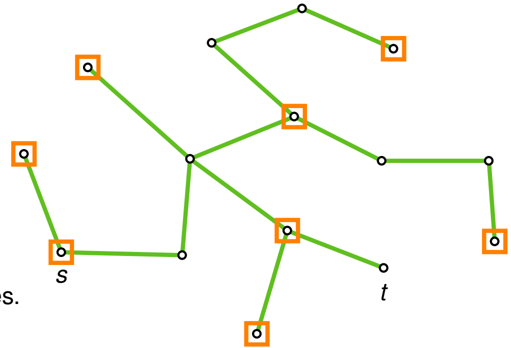

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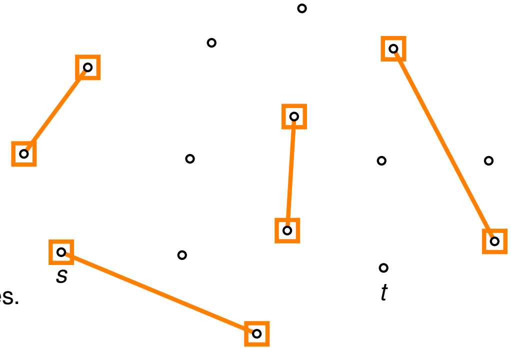

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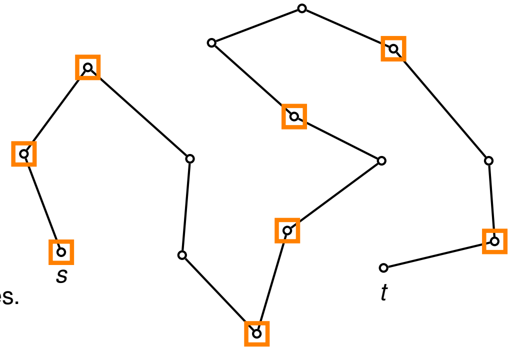

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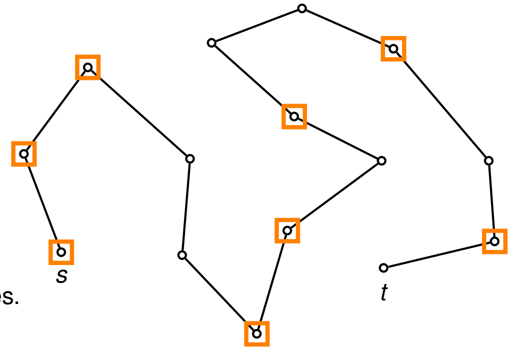

Goal: Find tree T with $\ell(T) \leqslant \ell(\mathrm{OPT})$ and s.t. shortest Q_T-join J satisfies $\ell(J) \leqslant \frac{1}{2} \cdot \ell(OPT)$.

- Held-Karp polytope for Path TSP:

$$P_{\mathrm{HK}} := \left\{ x \in \mathbb{R}^E_{\geqslant 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & v \in \{s,t\} \\ x(\delta(v)) = 2 & v \in V \setminus \{s,t\} \\ x(\delta(C)) \geqslant 1 & \forall C \subseteq V,\ |C \cap \{s,t\}| = 1 \\ x(\delta(C)) \geqslant 2 & \forall C \subsetneq V,\ C \neq \emptyset,\ |C \cap \{s,t\}| = 0 \end{array} \right\}$$

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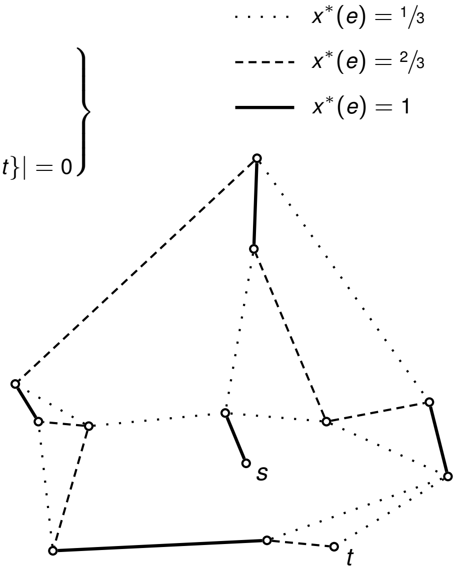

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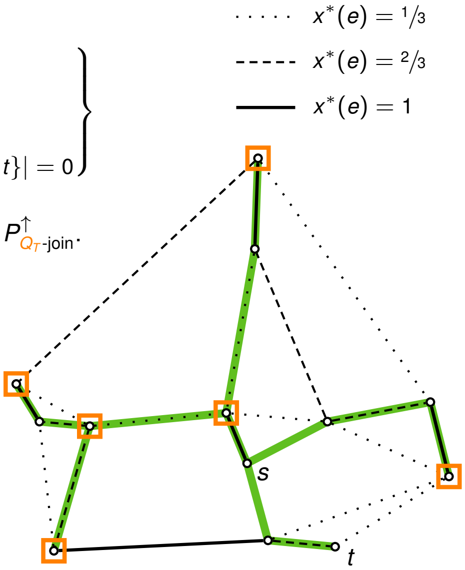
- Problem: $\frac{x^*}{2}$ for $x^* \in \operatorname{argmin}\{\ell^\top x \mid x \in P_{\mathrm{HK}}\}$ infeasible for $P^{\uparrow}_{Q_T\text{-join}}$.

$$P^{\uparrow}_{Q_T\text{-join}} = \left\{ x \in \mathbb{R}^E_{\geqslant 0} \;\middle|\; x(\delta(C)) \geqslant 1 \quad \begin{array}{l} \forall C \subseteq V, \\ |C \cap Q_T| \text{ odd} \end{array} \right\}$$



- Held-Karp polytope for Path TSP:

$$P_{\mathrm{HK}} := \left\{ x \in \mathbb{R}^E_{\geqslant 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & v \in \{s,t\} \\ x(\delta(v)) = 2 & v \in V \setminus \{s,t\} \\ x(\delta(C)) \geqslant 1 & \forall C \subseteq V,\ |C \cap \{s,t\}| = 1 \\ x(\delta(C)) \geqslant 2 & \forall C \subsetneq V,\ C \neq \emptyset,\ |C \cap \{s,t\}| = 0 \end{array} \right\}$$

- Problem: $\frac{x^*}{2}$ for $x^* \in \operatorname{argmin}\{\ell^\top x \mid x \in P_{\mathrm{HK}}\}$ infeasible for $P^{\uparrow}_{Q_T\text{-join}}$.

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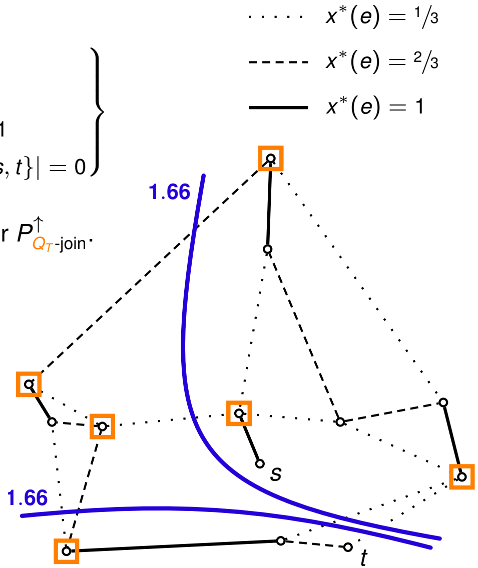
- Infeasibility caused by *narrow* cuts:
  - $\rightarrow$ cuts C with $x^*(\delta(C)) < 2$.
  - $\rightarrow$ s-t-cuts, form a chain.
  - $\rightarrow$ appear in $P^{\uparrow}_{Q_T\text{-join}}$ only if $|T \cap \delta(C)|$ even.



Where Wolsey's analysis fails

- Held-Karp polytope for Path TSP:

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- Infeasibility caused by *narrow* cuts:
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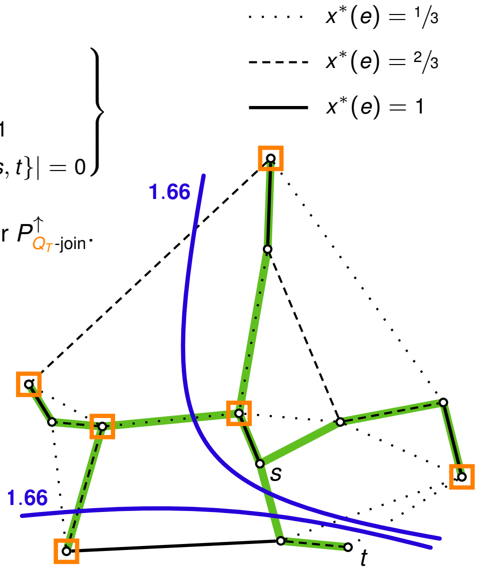
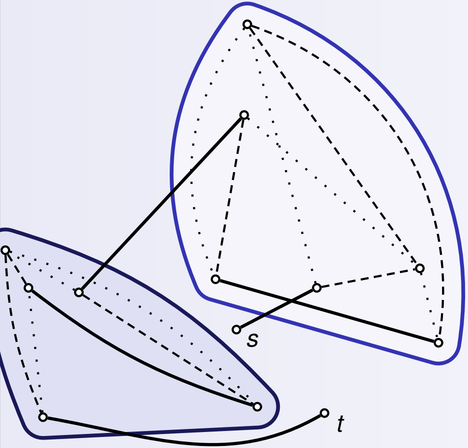




1.5-approximation: The high-level plan

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

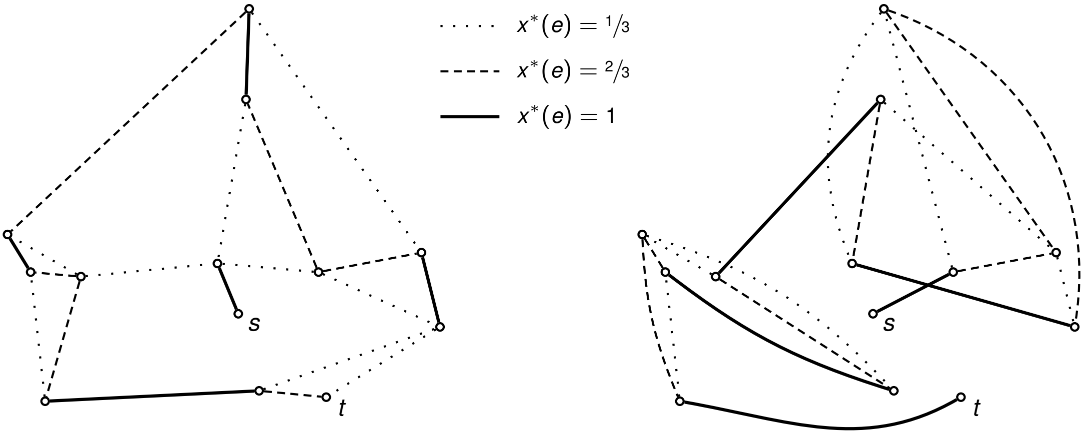
- Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
- Let
$$\mathcal{B}(x^*) := \{C \subseteq V \mid s \in C,\ t \notin C,\ x^*(\delta(C)) < 3\} \quad .$$
By Karger's result, $|\mathcal{B}(x^*)|$ is polynomially bounded. [Karger 1993]
- We will find a shortest point $y \in P_{\mathrm{HK}}$ that is $\mathcal{B}(x^*)$-good:

For each $B \in \mathcal{B}(x^*)$, either
- $y(\delta(B)) \geqslant 3$, or
- $y(\delta(B)) = 1$ and y is $0/1$ on $\delta(B)$.

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$\mathcal{B}(x^*)$-good

$y \in P_{\text{HK}}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geqslant 3$, or ▶ $y(\delta(B)) = 1$ and y is $0/1$ on $\delta(B)$.



$x^* \in P_{\text{HK}}$

$\mathcal{B}(x^*)$-good point y

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$\mathcal{B}(x^*)$-good

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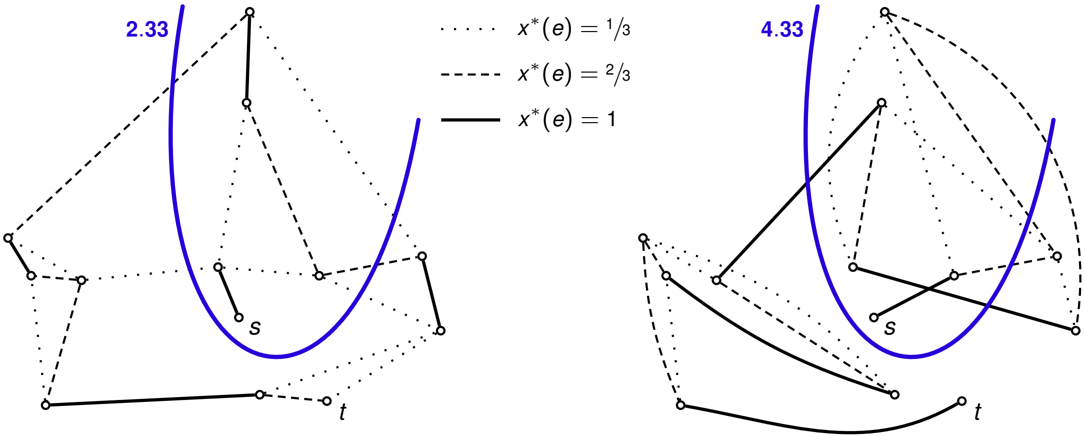

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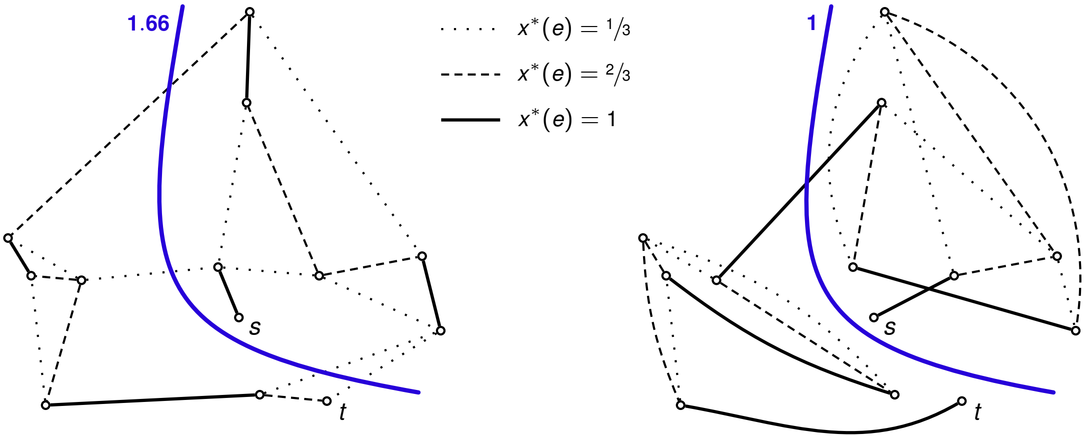

$x^* \in P_{\mathrm{HK}}$

$\mathcal{B}(x^*)$-good point y

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

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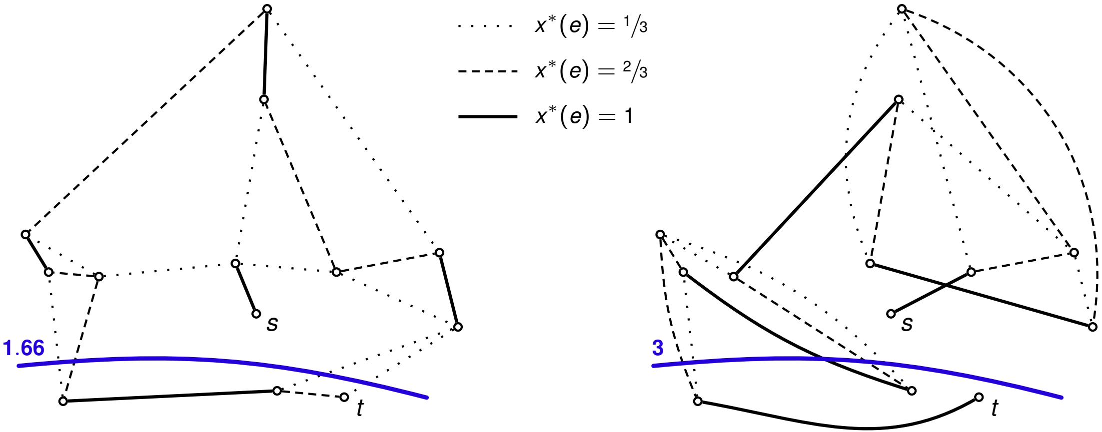

$x^* \in P_{\text{HK}}$

$\mathcal{B}(x^*)$-good point y

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$\mathcal{B}(x^*)$-good

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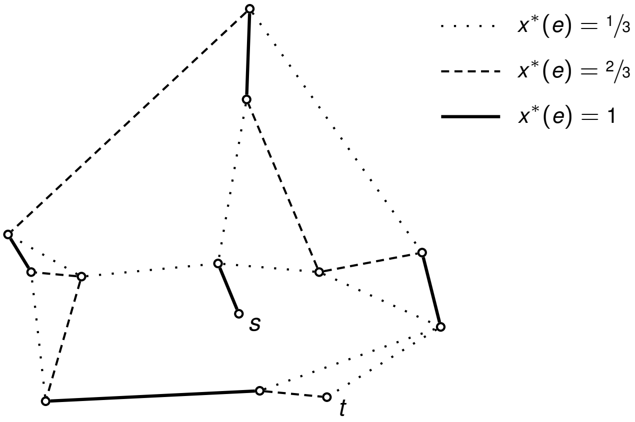
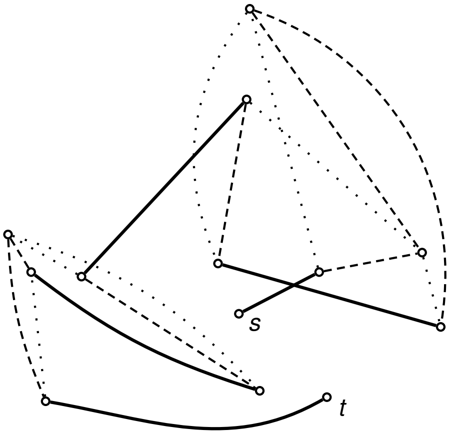


$x^* \in P_{\text{HK}}$



$\mathcal{B}(x^*)$-good point y

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

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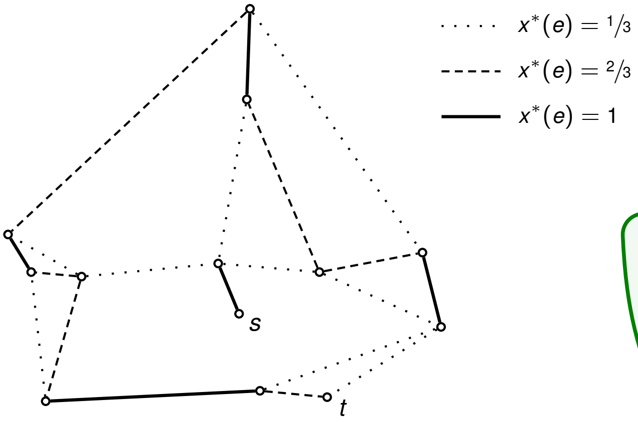
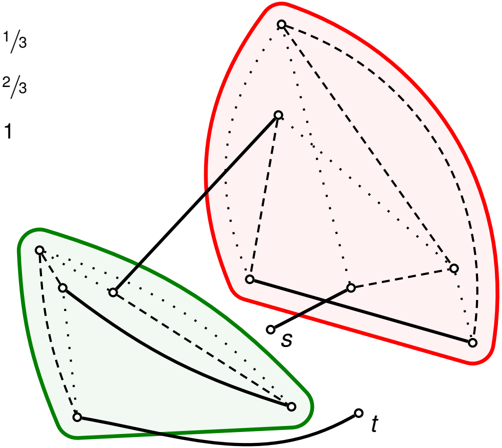


$x^* \in P_{\text{HK}}$



$\mathcal{B}(x^*)$-good point y

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

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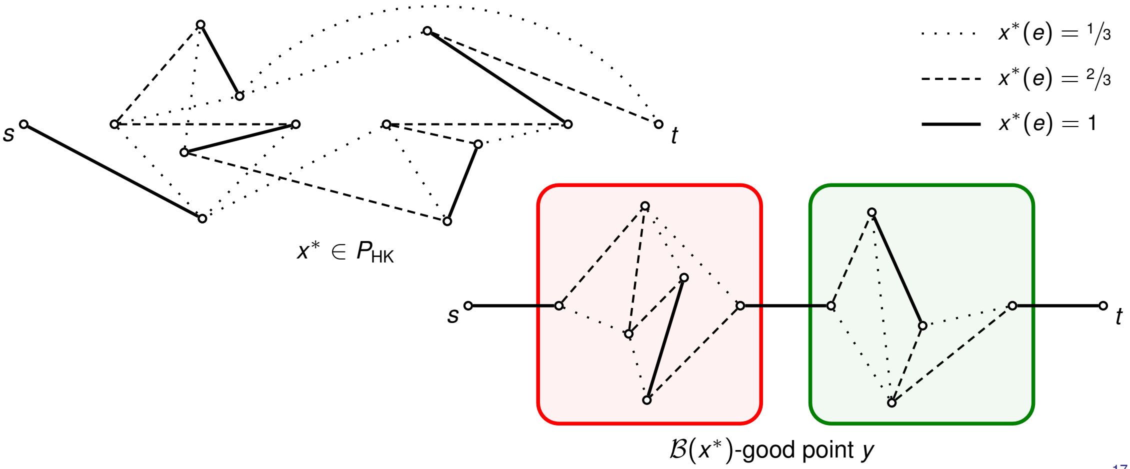

A new ingredient: Finding $\mathcal{B}(x^*)$-good points

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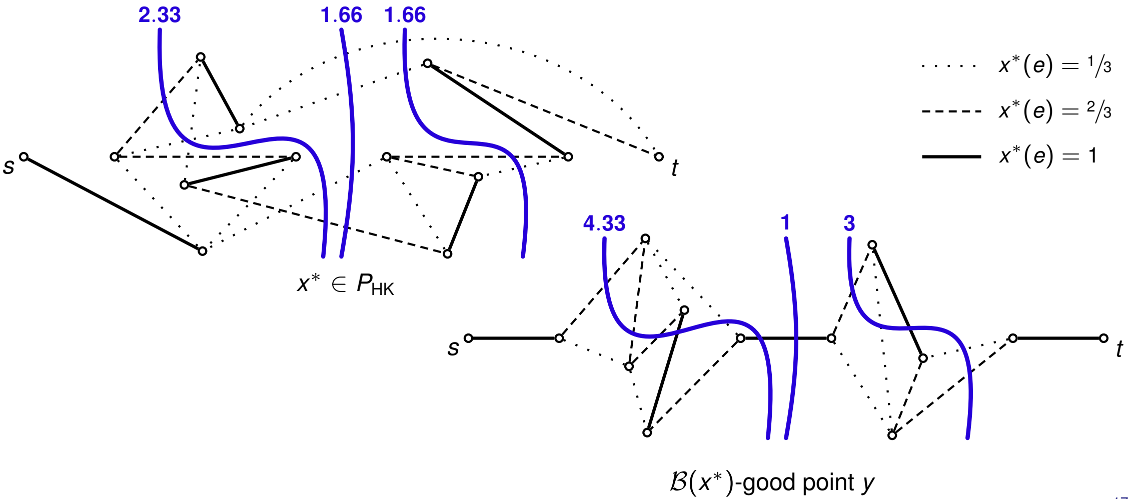

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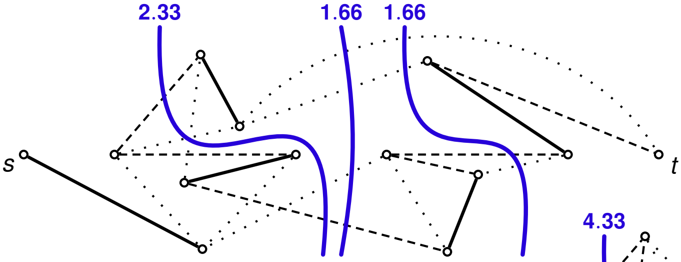
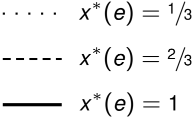
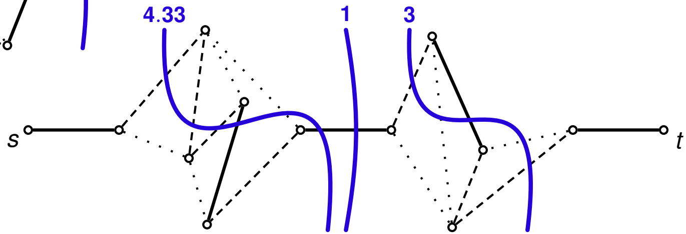




$x^* \in P_{\mathrm{HK}}$

Theorem

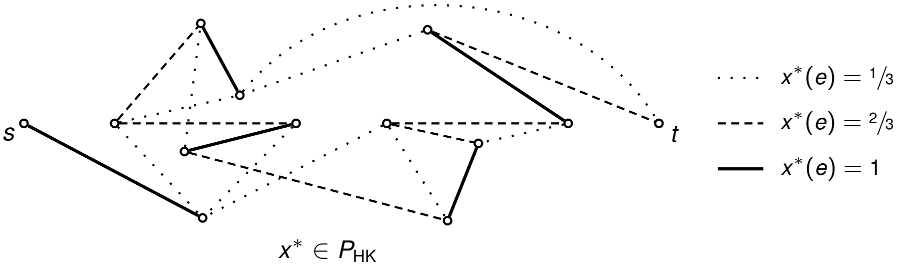
Let $\mathcal{B} \subseteq 2^V$ be family of s-t cuts. A shortest $\mathcal{B}$-good point $y \in P_{\mathrm{HK}}$ can be found in time $O(\mathrm{poly}(|V|, |\mathcal{B}|))$.



$\mathcal{B}(x^*)$-good point y

From short $\mathcal{B}(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
3. Let T be a shortest spanning tree in $(V, \operatorname{supp}(y))$.
4. Let J be a shortest Q_T-join.
5. Return shortcutted tour in multiunion of T and J.



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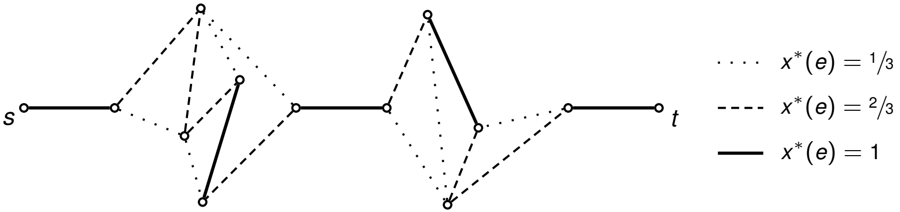

$\mathcal{B}(x^*)$-good $y \in P_{\mathrm{HK}}$

From short $\mathcal{B}(x^*)$-good points to 1.5-approx.

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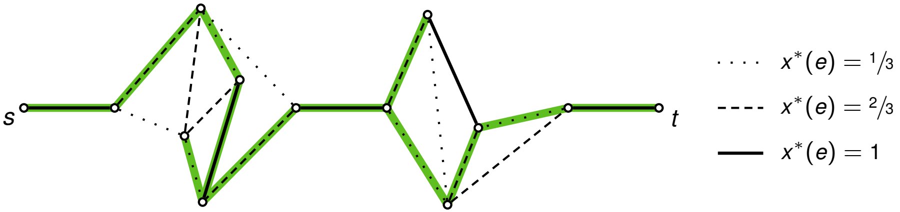

From short $\mathcal{B}(x^*)$-good points to 1.5-approx.

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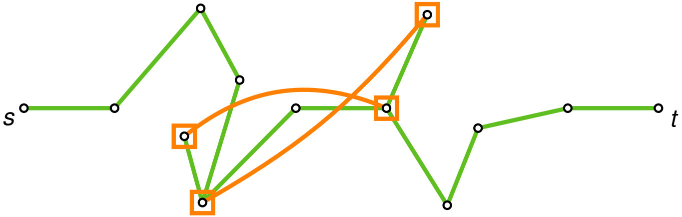

$\square$ $Q_T := \operatorname{odd}(T) \triangle \{s, t\}$

From short $\mathcal{B}(x^*)$-good points to 1.5-approx.

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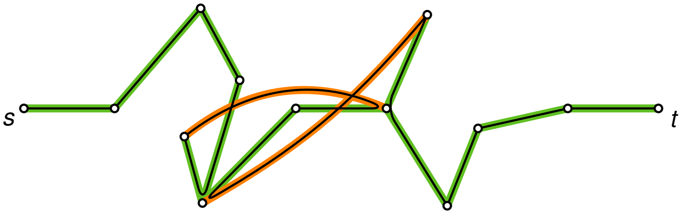

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1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\text{HK}}\}$.
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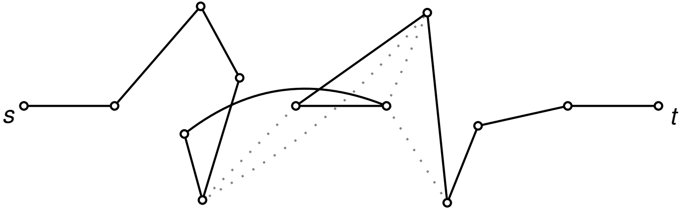

The spanning tree T is cheap: $\ell(T) \leqslant \ell(\mathrm{OPT})$

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
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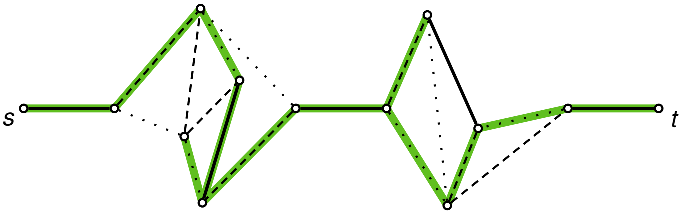

The spanning tree T is cheap: $\ell(T) \leqslant \ell(\mathrm{OPT})$

- We have $y \in P_{\mathrm{HK}} \subseteq P_{\mathrm{ST}}$.

  $$\implies \ell(\boldsymbol{T}) \leqslant \ell^{\top} y \ .$$

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
3. Let $\boldsymbol{T}$ be an MST in $(V, \operatorname{supp}(y))$.
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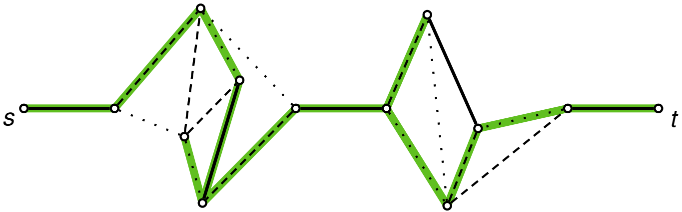

The spanning tree T is cheap: $\ell(T) \leqslant \ell(\mathrm{OPT})$

- We have $y \in P_{\mathrm{HK}} \subseteq P_{\mathrm{ST}}$.

$$\implies \ell(\boldsymbol{T}) \leqslant \ell^\top y \ .$$

- OPT is $\mathcal{B}$-good for any family $\mathcal{B}$ of s-t cuts.

$$\implies \ell^\top y \leqslant \ell(\mathrm{OPT}) \ .$$

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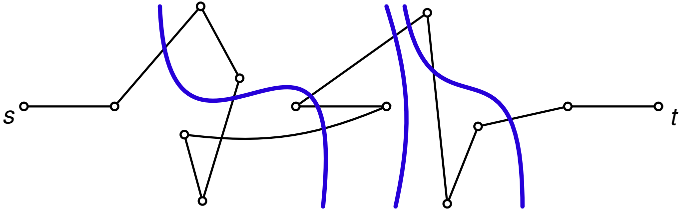

- We have $y \in P_{\mathrm{HK}} \subseteq P_{\mathrm{ST}}$.

$$\implies \ell(T) \leqslant \ell^\top y \ .$$

- OPT is $\mathcal{B}$-good for any family $\mathcal{B}$ of s-t cuts.

$$\implies \ell^\top y \leqslant \ell(\mathrm{OPT}) \ .$$

- Together, we conclude

$$\ell(T) \leqslant \ell^\top y \leqslant \ell(\mathrm{OPT}) \ .$$

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
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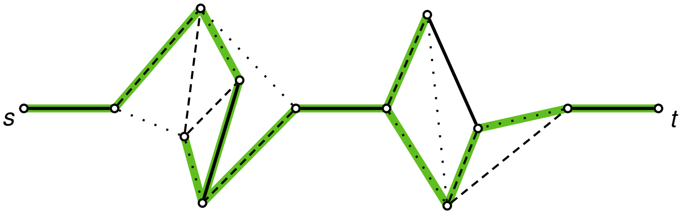

The Q_T-join J is cheap: $\ell(J) \leqslant \frac{1}{2}\,\ell(\mathrm{OPT})$

- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P^{\uparrow}_{Q_T\text{-join}}$.

$$\implies \ell(J) \leqslant \frac{1}{4}\left(\ell^\top x^* + \ell^\top y\right) \leqslant \frac{1}{2}\ell(\mathrm{OPT}) \;.$$

- Distinguish cases:

1. 2. 3. 4.

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
3. Let T be an MST in $(V, \operatorname{supp}(y))$.
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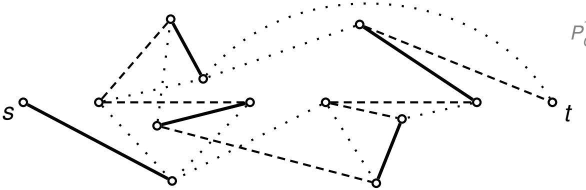
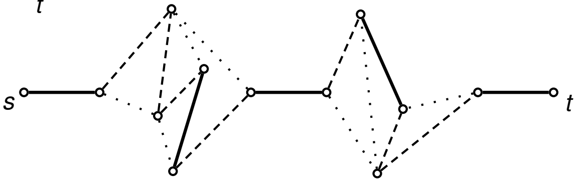
5. Return shortcutted tour in multiunion of T and J.

$$P^{\uparrow}_{Q_T\text{-join}} = \left\{x \in \mathbb{R}^E_{\geqslant 0} \;\middle|\; x(\delta(C)) \geqslant 1 \quad \forall C \subseteq V,\ |C \cap Q_T| \text{ odd}\right\}$$



$x^* \in P_{\mathrm{HK}}$



$\mathcal{B}(x^*)$-good point y

The Q_T-join J is cheap: $\ell(J) \leqslant \frac{1}{2}\,\ell(\mathrm{OPT})$

- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P^{\uparrow}_{Q_T\text{-join}}$.

$$\implies \ell(J) \leqslant \frac{1}{4}\left(\ell^\top x^* + \ell^\top y\right) \leqslant \frac{1}{2}\ell(\mathrm{OPT}) \;.$$

- Distinguish cases:

1. Non s-t cuts. **2.** **3.** **4.**

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
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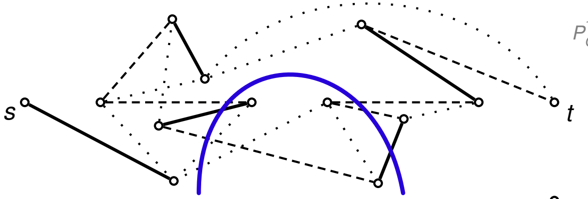
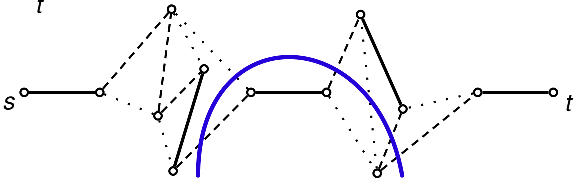


$x^* \in P_{\mathrm{HK}}$

$x^*(\delta(\boldsymbol{B})) \geqslant 2 \qquad y(\delta(\boldsymbol{B})) \geqslant 2$



$\mathcal{B}(x^*)$-good point y

The Q_T-join J is cheap: $\ell(J) \leqslant \frac{1}{2}\,\ell(\mathrm{OPT})$

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
3. Let T be an MST in $(V, \operatorname{supp}(y))$.
4. Let J be a shortest Q_T-join.
5. Return shortcutted tour in multiunion of T and J.

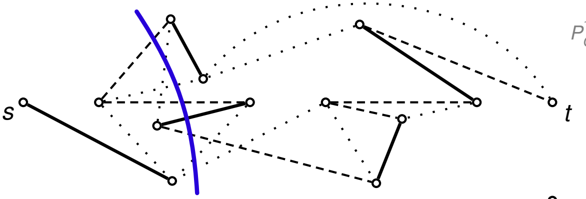
- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P^{\uparrow}_{Q_T\text{-join}}$.

$$\Longrightarrow \ \ell(J) \leqslant \frac{1}{4}\left(\ell^\top x^* + \ell^\top y\right) \leqslant \frac{1}{2}\ell(\mathrm{OPT}) \ .$$

$$P^{\uparrow}_{Q_T\text{-join}} = \left\{x \in \mathbb{R}^E_{\geqslant 0} \;\middle|\; x(\delta(C)) \geqslant 1 \quad \forall C \subseteq V,\ |C \cap Q_T| \text{ odd}\right\}$$

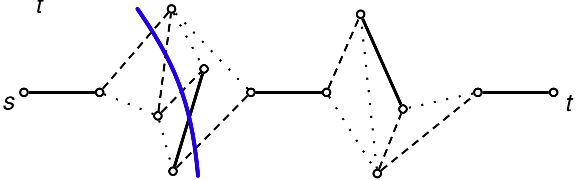
- Distinguish cases:

1. | **2.** s-t cuts not in $\mathcal{B}(x^*)$. | **3.** | **4.**



$x^* \in P_{\mathrm{HK}}$

$x^*(\delta(\boldsymbol{B})) \geqslant 3$ $\qquad$ $y(\delta(\boldsymbol{B})) \geqslant 1$



$\mathcal{B}(x^*)$-good point y

The Q_T-join J is cheap: $\ell(J) \leqslant \frac{1}{2}\,\ell(\mathrm{OPT})$

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- Distinguish cases:

1. **2.** **3.** s-t cuts $B \in \mathcal{B}(x^*)$ with $y(\delta(B)) \geqslant 3$. **4.**

1. Let $x^* \in \operatorname{argmin}\{\ell(x) \mid x \in P_{\mathrm{HK}}\}$.
2. Let y be a shortest $\mathcal{B}(x^*)$-good point.
3. Let T be an MST in $(V, \operatorname{supp}(y))$.
4. Let J be a shortest Q_T-join.
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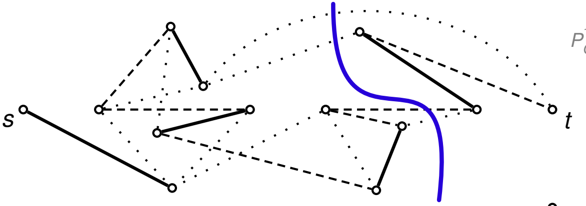

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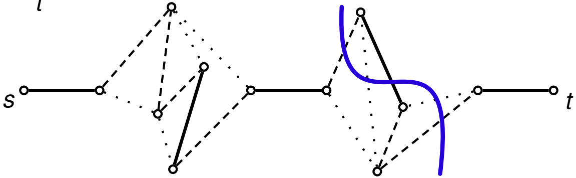

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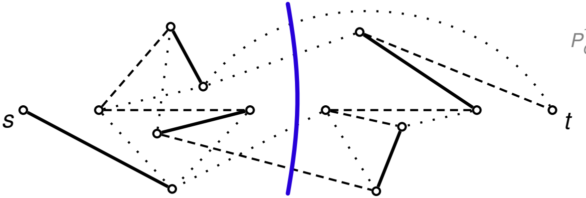

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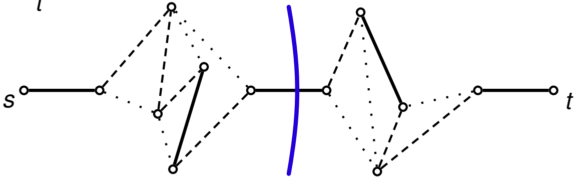

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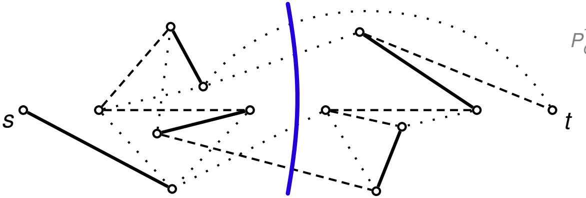

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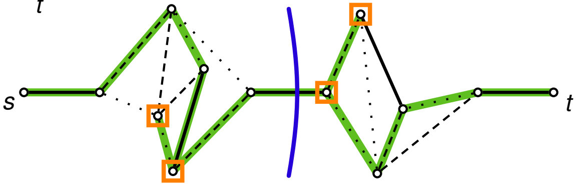
$|B \cap Q_T|$ even



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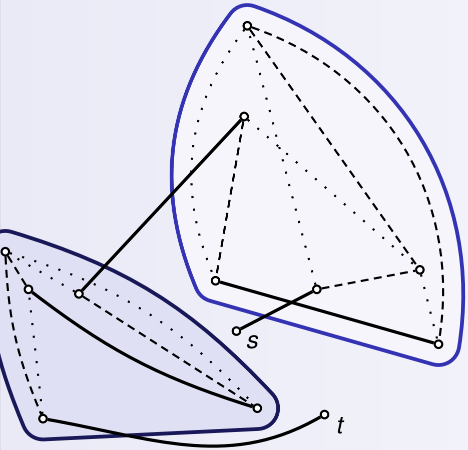
The dynamic program



The DP: Finding shortest $\mathcal{B}$-good points

Theorem

Let $\mathcal{B} \subseteq 2^V$ a family of s-t cuts. A shortest $\mathcal{B}$-good point $y \in P_{\mathrm{HK}}$ can be found in time $O(\mathrm{poly}(|V|, |\mathcal{B}|))$.

$\mathcal{B}$-good point y

For all $B \in \mathcal{B}$, either

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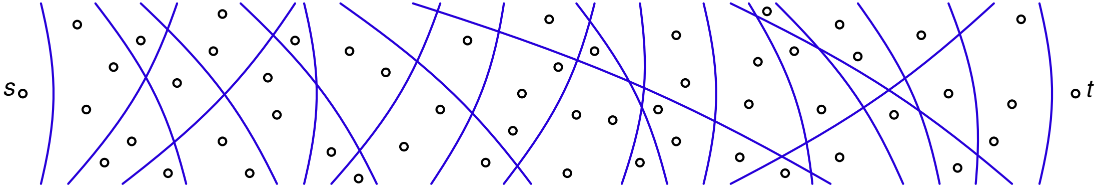

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- Key plan:
  - "Guess" cuts $B_1, \dots, B_k \in \mathcal{B}$ with $y(\delta(B_i)) = 1$, and the single edge in these cuts.
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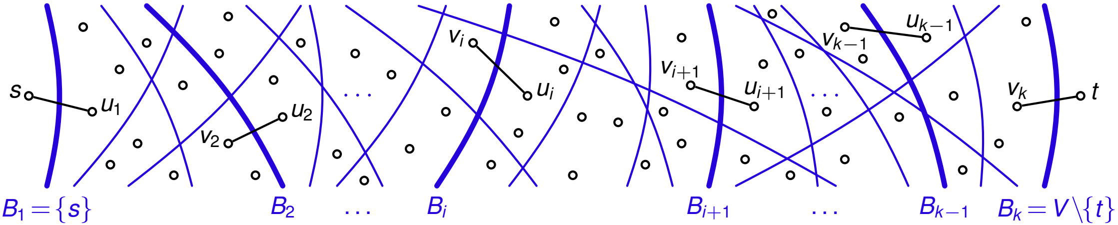

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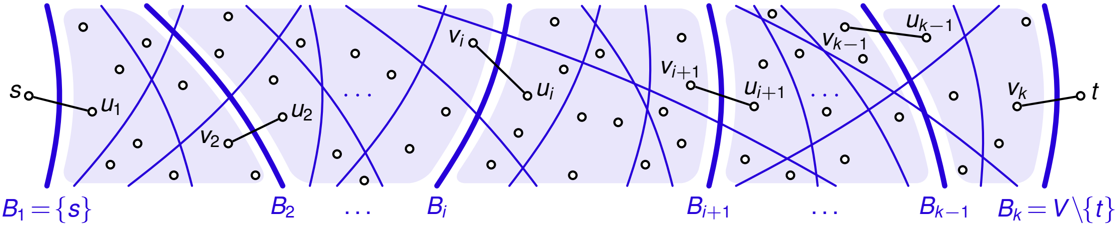
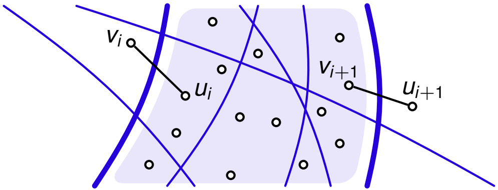


- Restriction to $B_{i+1} \setminus B_i$, start at u_i, end at v_{i+1}.
- Enforce $y(\delta(B)) \geqslant 3$ for $B \in \mathcal{B}$ with $B_i \subsetneq B \subsetneq B_{i+1}$.
- Corresponding LP formulation:

$$\begin{aligned}\lambda(B_{i+1} \setminus B_i, u_i, v_{i+1}) = \min\ & \ell^\top y \\ & y \in P_{\mathrm{HK}}(B_{i+1} \setminus B_i, u_i, v_{i+1}) \\ & y(\delta(B)) \geqslant 3 \qquad \forall B \in \mathcal{B} \colon B_i \subsetneq B \subsetneq B_{i+1} \ .\end{aligned}$$



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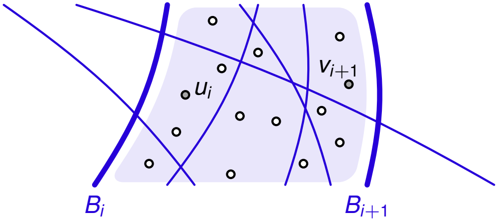

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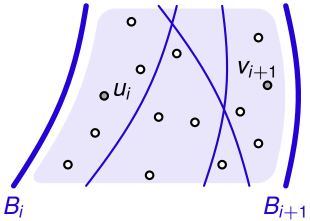

- Idea: Advance from one cut B with $y(\delta(B)) = 1$ to another.

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- Formulation as a shortest path problem on auxiliary digraph:

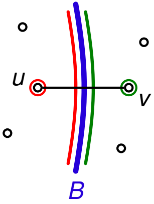
Nodes: Pairs (B, v) for $B \in \mathcal{B}$ and $v \in V$.

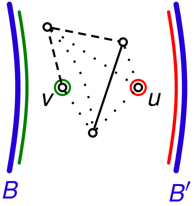
Edges: Two types of steps corresponding to extension of a solution.

$$(B, u) \xrightarrow{\ell((u,v))} (B, v)$$

$$(B, v) \xrightarrow{\lambda(B' \setminus B, v, u)} (B', u)$$

$u \in B$
$v \in V \setminus B$





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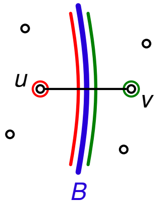

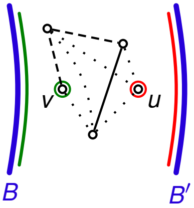

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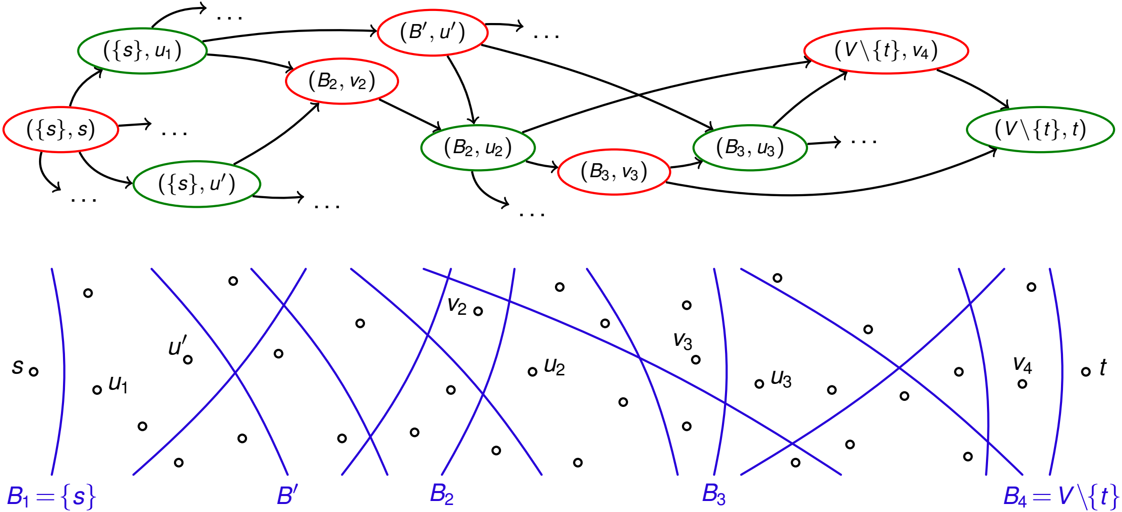
- Optimal solution: Shortest $(\{s\}, s)$-$(V \setminus \{t\}, t)$ path in auxiliary digraph.

DP auxiliary graph: An example



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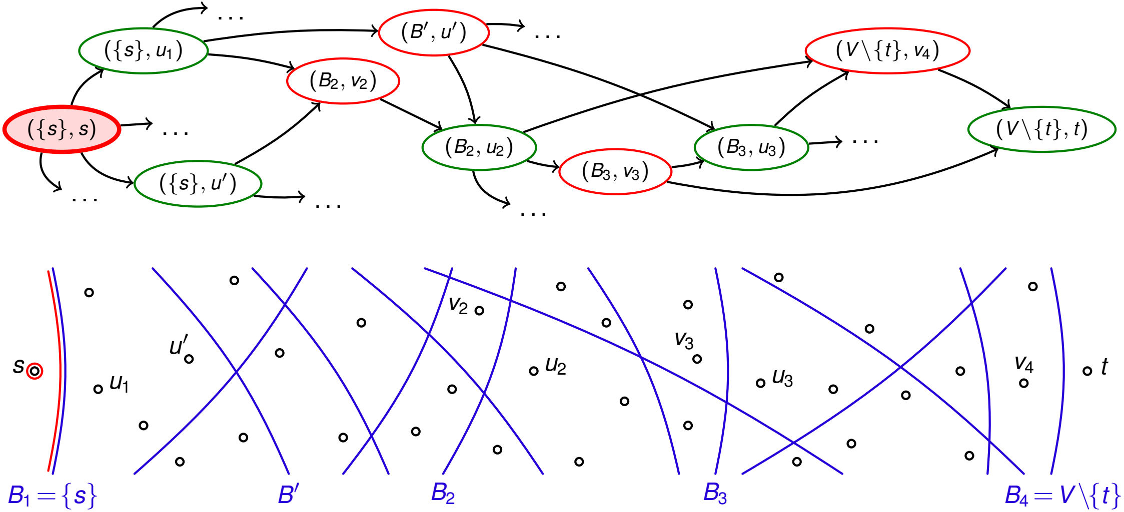
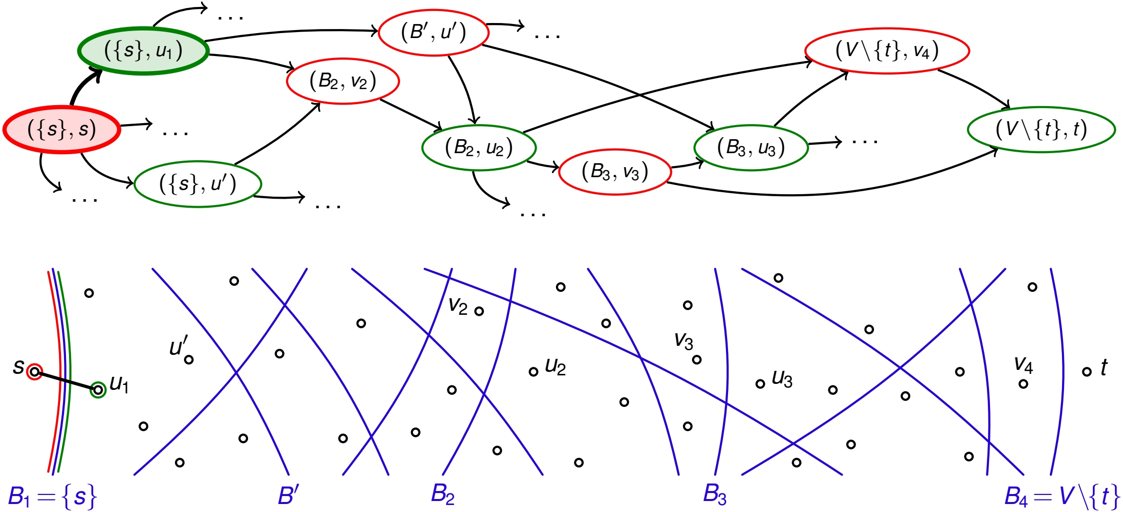
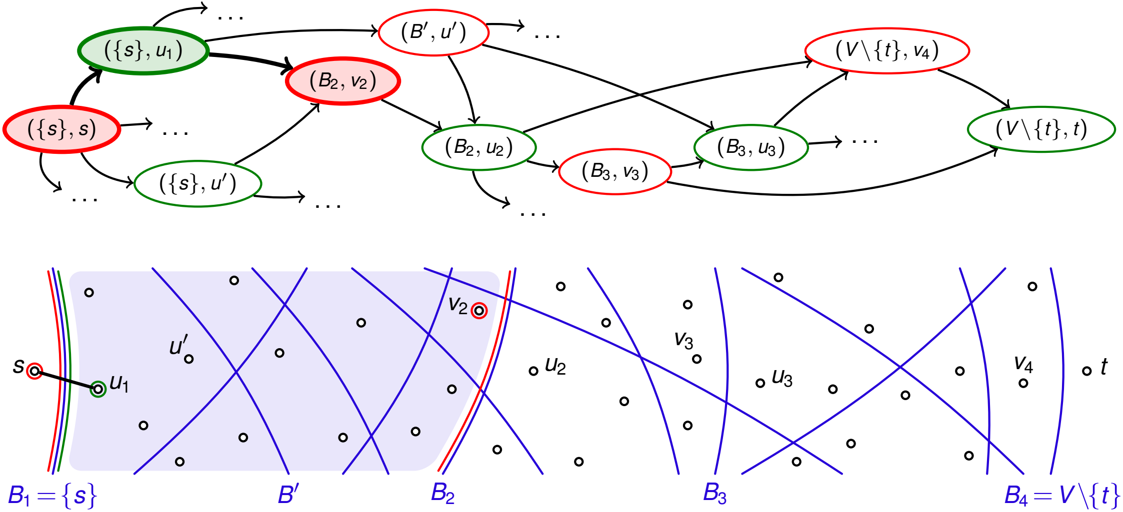



$(\{s\}, u_1)$
$(\{s\}, s)$
$(\{s\}, u')$
(B', u')
(B_2, v_2)
(B_2, u_2)
(B_3, v_3)
(B_3, u_3)
$(V \setminus \{t\}, v_4)$
$(V \setminus \{t\}, t)$
$\dots$
s
u_1
u'
v_2
u_2
v_3
u_3
v_4
t
$B_1 = \{s\}$
B'
B_2
B_3
$B_4 = V \setminus \{t\}$



$(\{s\}, u_1)$
(B_2, v_2)
$(\{s\}, s)$
$(\{s\}, u')$
(B', u')
(B_2, u_2)
(B_3, v_3)
(B_3, u_3)
$(V\setminus\{t\}, v_4)$
$(V\setminus\{t\}, t)$
s
u_1
u'
v_2
u_2
v_3
u_3
v_4
t
$B_1 = \{s\}$
B'
B_2
B_3
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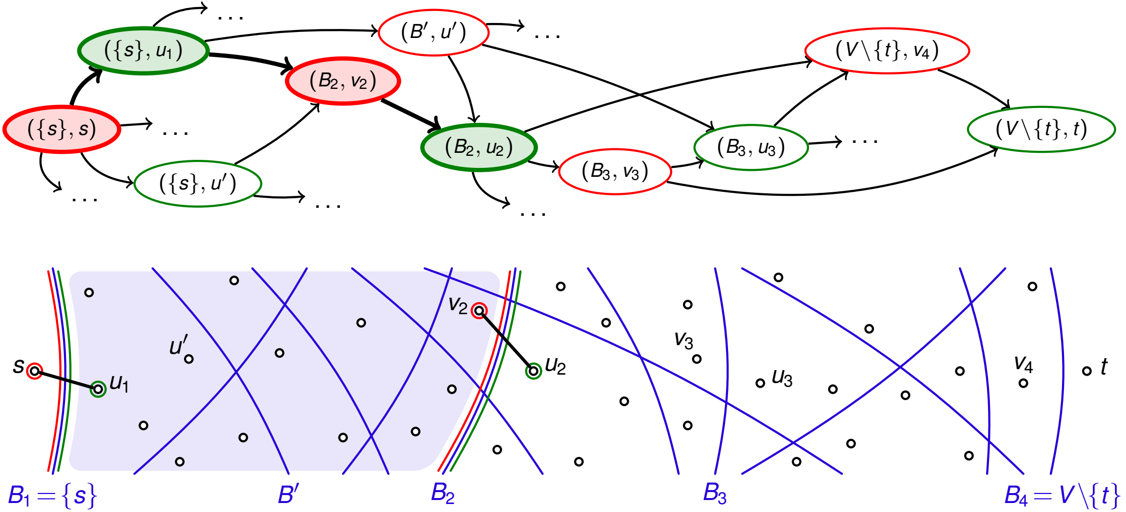

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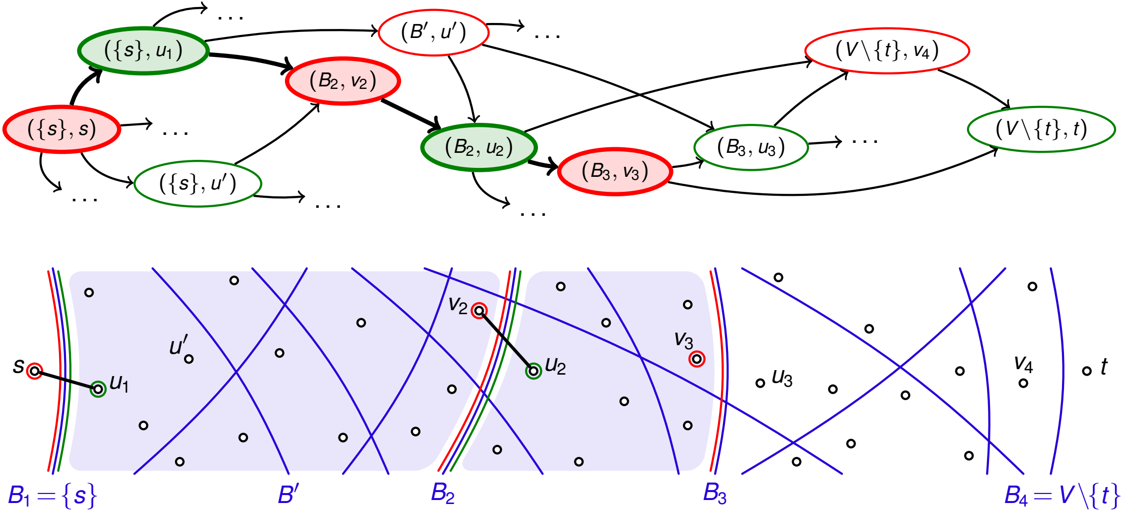

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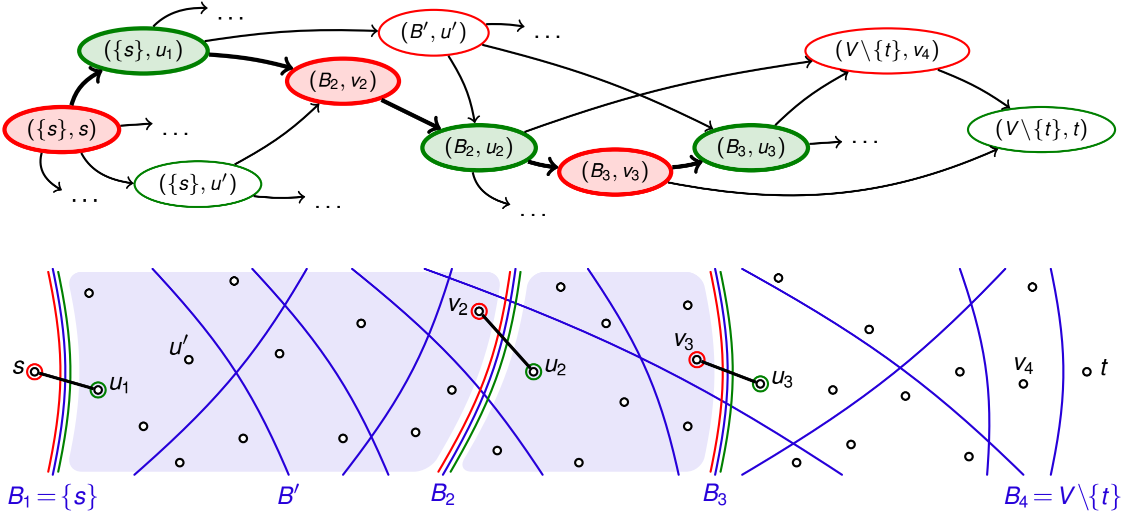

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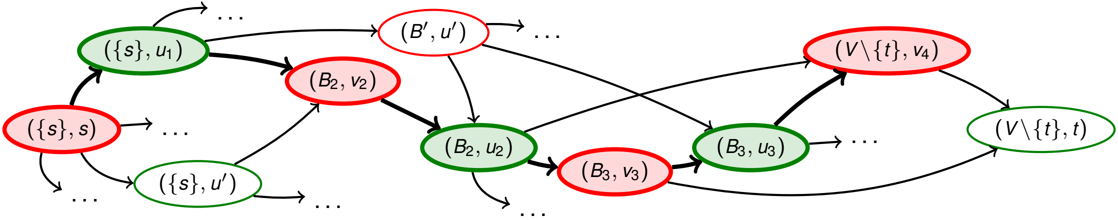

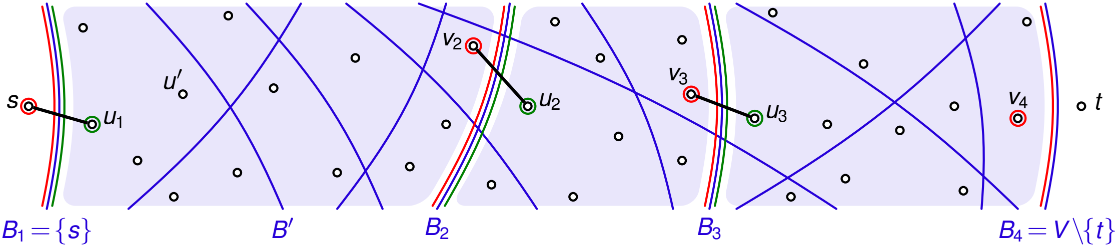

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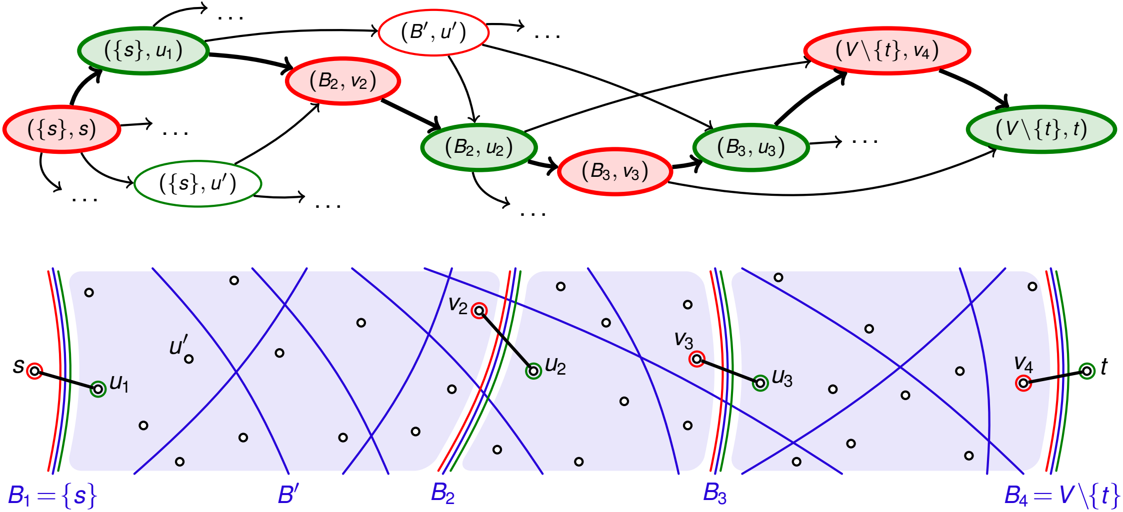

- DP solution: Combination of edges in cuts B_i and partial Held-Karp solutions.

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Theorem (basic properties of DP solutions)

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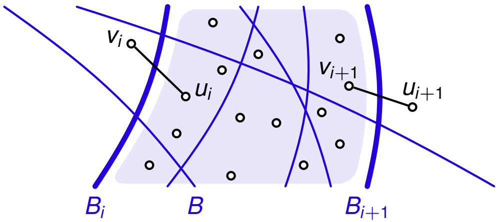

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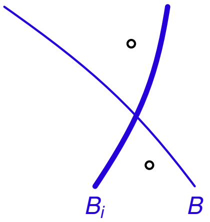

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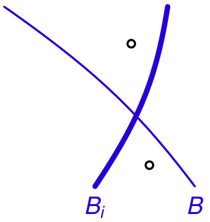

$$y(\delta(B)) + y(\delta(B_i))$$

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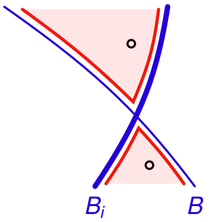

$$y(\delta(B)) + y(\delta(B_i)) \geqslant y(\delta(B_i \setminus B)) + y(\delta(B \setminus B_i))$$

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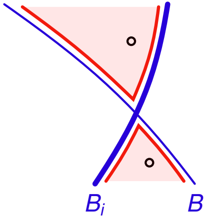

$$y(\delta(B)) + \underbrace{y(\delta(B_i))}_{=1} \geqslant \underbrace{y(\delta(B_i \setminus B))}_{\geqslant 2} + \underbrace{y(\delta(B \setminus B_i))}_{\geqslant 2}$$

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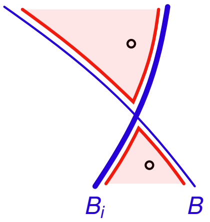
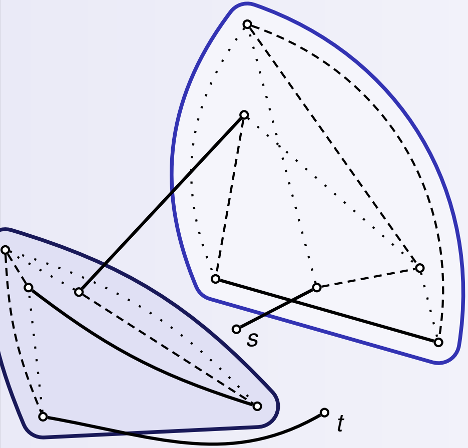
- Shortest DP solution has length at most $\ell(\mathrm{OPT})$.

Proof. Need to show that DP solution y satisfies $y(\delta(B)) \geqslant 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$.



$$y(\delta(B)) + \underbrace{y(\delta(B_i))}_{=1} \geqslant \underbrace{y(\delta(B_i \setminus B))}_{\geqslant 2} + \underbrace{y(\delta(B \setminus B_i))}_{\geqslant 2}$$

$$\Longrightarrow \quad y(\delta(B)) \geqslant 3 \; .$$



Conclusions

Theorem **[Zenklusen, 2018]**

There is a 1.5-approximation for Path TSP.

- Approximation factors below 1.5 for TSP (or even Path TSP)?
- Show that the integrality gap of Held-Karp relaxation for Path TSP is 1.5.

  Current best: 1.5284. [Traub, Vygen, 2018c]
- 1.5-approximation for T-tours?

  True if $|T| = O(1)$. [N., Zenklusen, 2019]