Congruency-Constrained Optimization

— at the interface of Integer Programming & Combinatorial Optimization —

Martin Nägele

Research Institute for Discrete Mathematics & HCM

University of Bonn

Based on joint past and ongoing work with
Ch. Nöbel, R. Santiago, B. Sudakov, and R. Zenklusen.

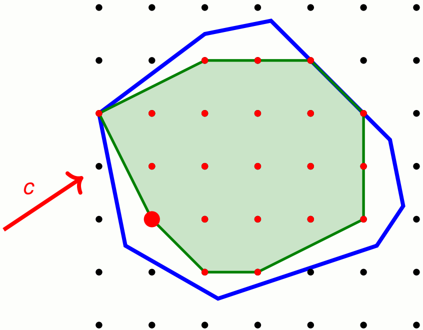
Swiss National Science Foundation

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Motivation & Background

bounded subdeterminant IPs — successes
in the bimodular case — new results

Towards general classes of efficiently solvable IPs



Integer Linear Programming (IP)

Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve
$$\min\{c^\top x \colon Ax \leqslant b,\ x \in \mathbb{Z}^n\}\ .$$

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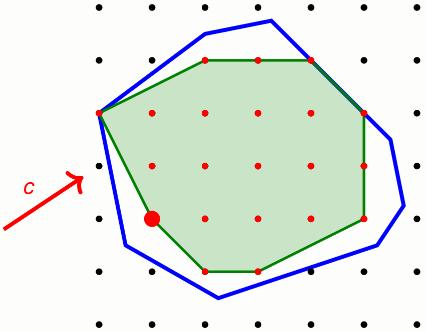

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An interesting class of efficiently solvable IPs

A totally unimodular (TU) $\implies$ Integral relaxation.

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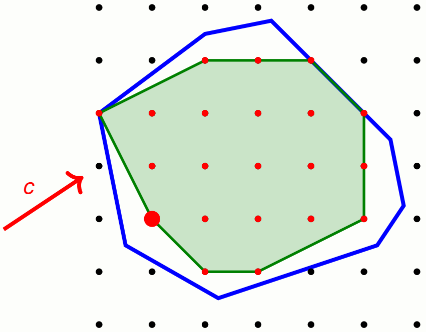

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What if minors, in absolute value, are still bounded, but not by 1?

Bounded subdeterminants

Δ-modular Integer Programming

Given a constant $\Delta \in \mathbb{Z}_{>0}$, can integer linear programs

$$\min\{c^\top x \colon Ax \leq b,\ x \in \mathbb{Z}^n\}$$

with Δ-modular constraint matrix A be solved efficiently?

- $A \in \mathbb{Z}^{m \times n}$ is Δ-modular if
  - → $\operatorname{rank}(A) = n$, and
  - → absolute values of $n \times n$ subdeterminants are bounded by Δ
- Δ-modularity is more general than *total* Δ-modularity

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Known results

✓ $\Delta = 1$: easy

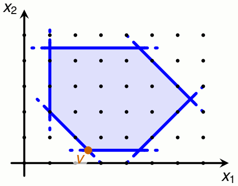
✓ $\Delta = 2$: Bimodular Integer Programming (BIP)

[Artmann, Weismantel, and Zenklusen, STOC 2017]

✓ Arbitrary constant Δ, at most 2 non-zeros per row

[Fiorini, Joret, Weltge, and Yuditsky, FOCS 2021]

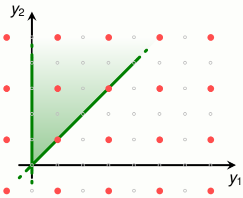
The approach to BIP



Bimodular Integer Program (BIP)

$$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$$

A bimodular.



Conic Parity TU Problem (CPTU)

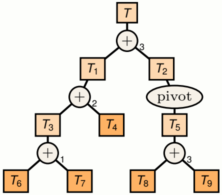
$$\min\{\tilde{c}^\top y \colon Ty \leqslant 0, y \in \mathbb{Z}^n, y(S) \text{ odd}\}$$

T totally unimodular, $S \subseteq [n]$.

Theorem

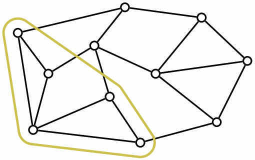
BIP can be solved in strongly polynomial time.

[Artmann, Weismantel, and Zenklusen, STOC 2017]



Seymour's TU decomposition

Reduction to base block problems.



Base block problems

Interpretation as parity-constrained cut and circulation problems.

Challenge: Generalize!

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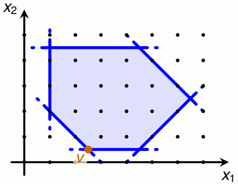

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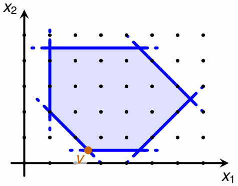

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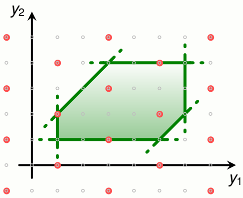

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$$\min\left\{\tilde{c}^\top y \colon \begin{array}{c} Ty \leqslant b,\ y \in \mathbb{Z}^n, \\ \gamma^\top y \equiv r \pmod{m} \end{array}\right\}$$

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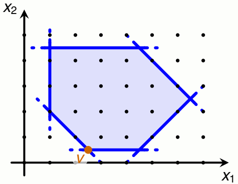

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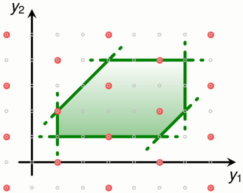

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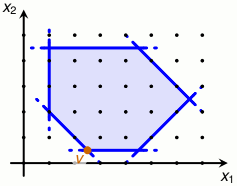

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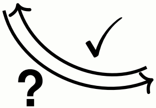

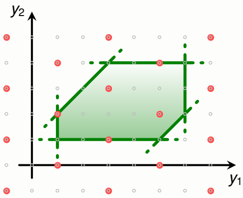

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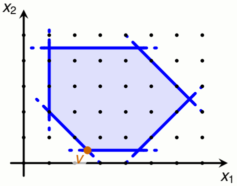

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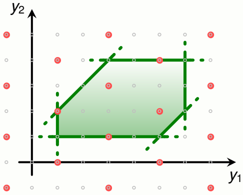

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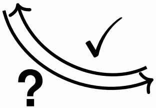

Equivalence:
strictly Δ-modular IP
(subdets in $\{0, \pm\Delta\}$)
if Δ prime

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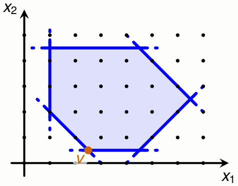

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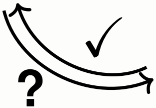

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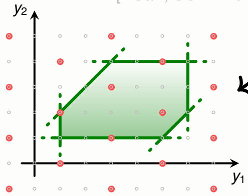
$\rightarrow$ proximity
$\rightarrow$ flatness or feasibility
[NSaZ, SODA 2022]



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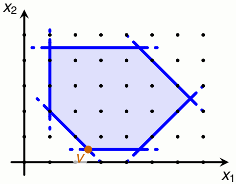

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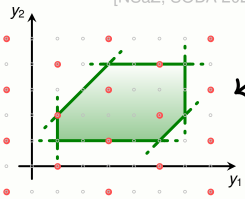

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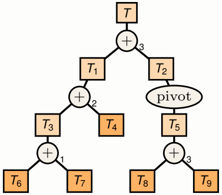

Seymour's TU decomposition

Reduction gets much more involved!

Generalization: $m = 3$, feasibility

→ hierarchy of problems

→ powerful tools, e.g., Cauchy-Davenport

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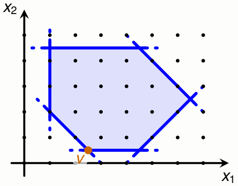

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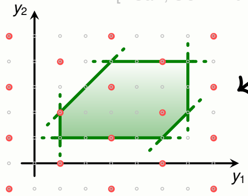

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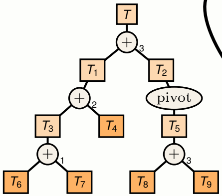

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[NNSaZ, 2022+]

Circulation baseblock:

$\rightarrow$ rand. alg. for unary enc. obj.

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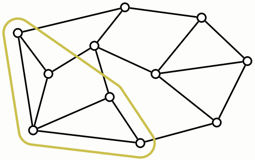

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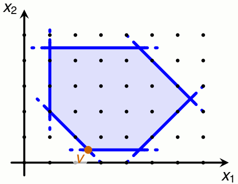

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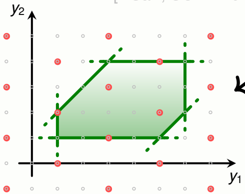

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Corollary

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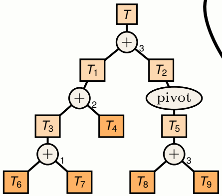

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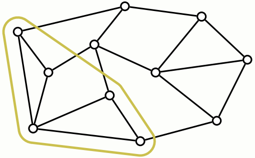

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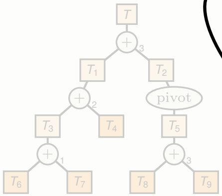

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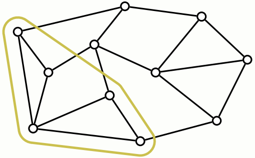

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Base Block Problems

Seymour's decomposition — network matrices — the two cases

Seymour's decomposition

Theorem: Seymour's decomposition [Seymour, 1980]

For every TU matrix $T \in \mathbb{Z}^{k \times n}$, one of the following applies:

(i) T is, possibly after row/column permutations and a pivot, of the form

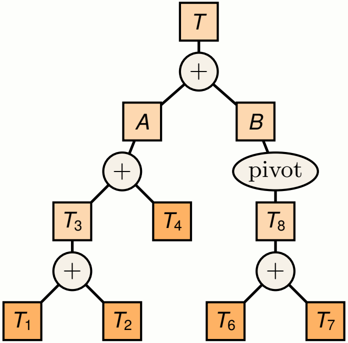
$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix},$$

where $\begin{pmatrix} A & e & e \\ h^\top & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} B & g & g \\ f^\top & 0 & 1 \end{pmatrix}$ are TU.

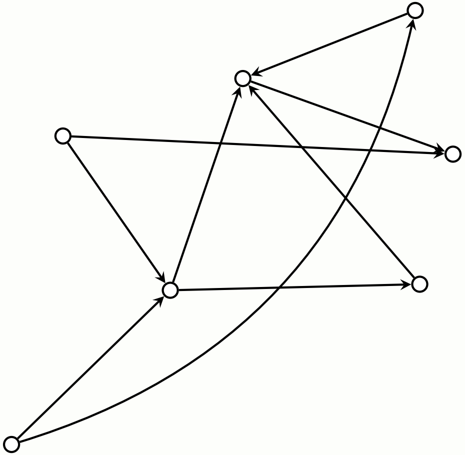
(ii) T is essentially equal to one of

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(iii) T or $T^\top$ is a network matrix.

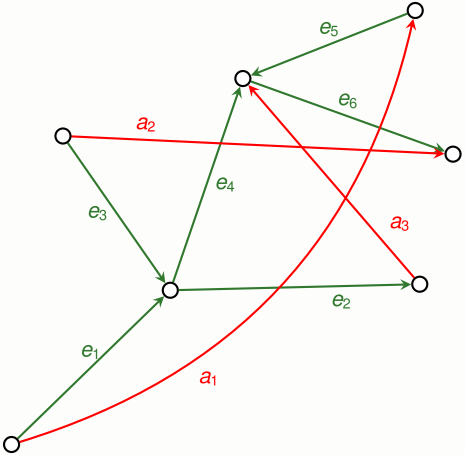


Network matrices



$$\left\{ \begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right\}$$

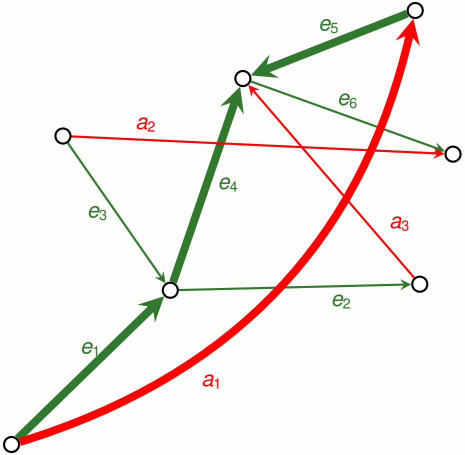
Network matrices



Tree $T = (V, E)$, extra arcs A.

$$
\begin{array}{c c}
 & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} &
\left\{ \begin{array}{rrr}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
-1 & 0 & 0 \\
0 & 1 & 0
\end{array} \right\}
\end{array}
$$

Network matrices



Tree $T = (V, E)$, extra arcs A.

	a_1	a_2	a_3
e_1	1	0	0
e_2	0	0	-1
e_3	0	1	0
e_4	1	1	1
e_5	-1	0	0
e_6	0	1	0

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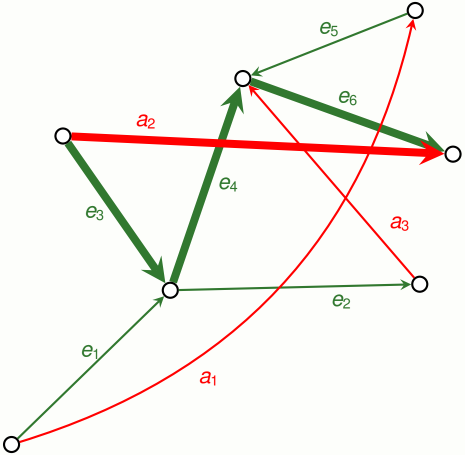

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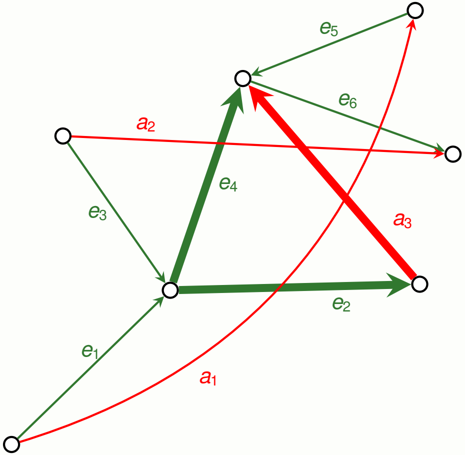

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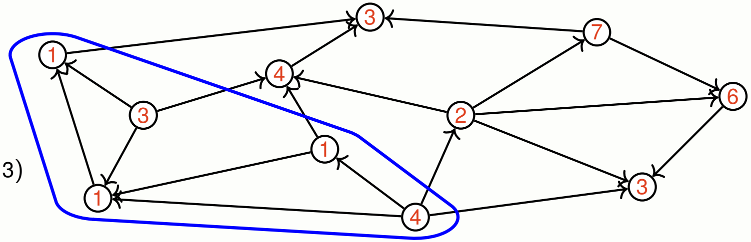
Congruency-constrained min (di-)cut

$$\min_{C \subsetneq V}\{|\delta^+(C)| \colon \delta^-(C) = \emptyset, \gamma(C) \equiv r \pmod{m}\}$$

on digraph $G = (V, A)$ with $\gamma \colon V \to \mathbb{Z}$.

$\gamma(C) \equiv 1 \pmod{3}$



Prime power moduli

Theorem

Congruency-constrained min-cut is poly-time solvable for constant prime power m.

[NSuZ, SODA 2018]

Algorithm

- Guess $m-1$ elements in- and outside OPT
- Solve corresp. unconstrained min cut problem
- Return best cong-constraint feasible solution

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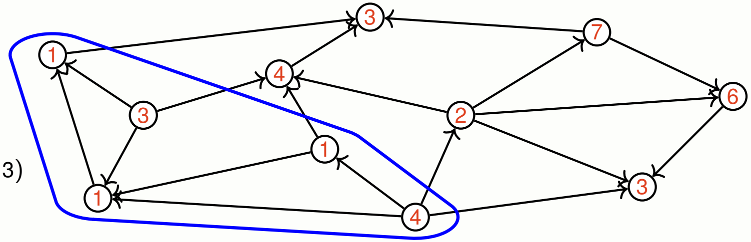

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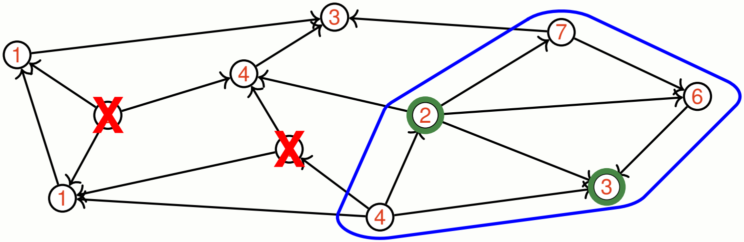

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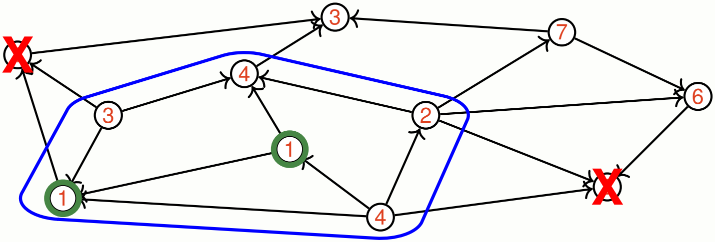

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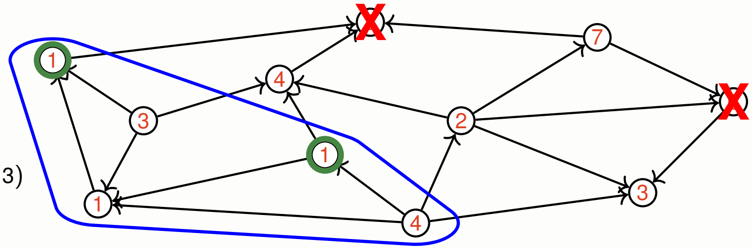

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Proof idea:

- Failing guesses give structured set system.
- Such systems cannot exist for prime power m.

The feasibility problem

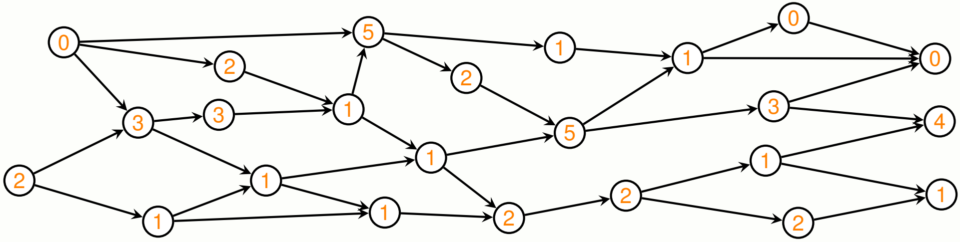
Congruency-constrained lattice feasibility

In $G = (V, A)$ with $\gamma \colon V \to \mathbb{Z}$, find
$C \subsetneq V \colon \delta^-(C) = \emptyset, \gamma(C) \equiv r \pmod{m}$.

Theorem

Congruency-constr. lattice feasibility can be decided in poly time for constant m.

[NNSaZ, 2022+]



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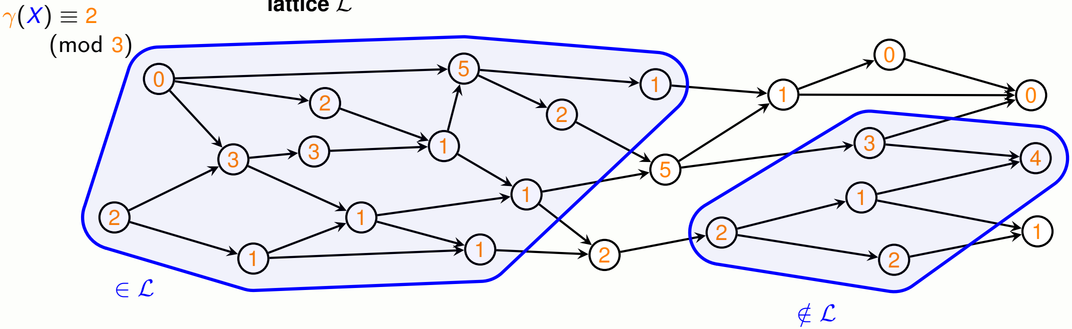

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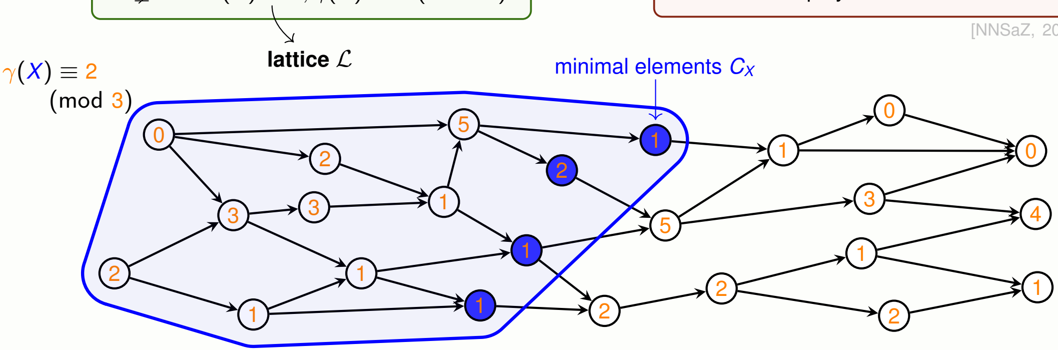

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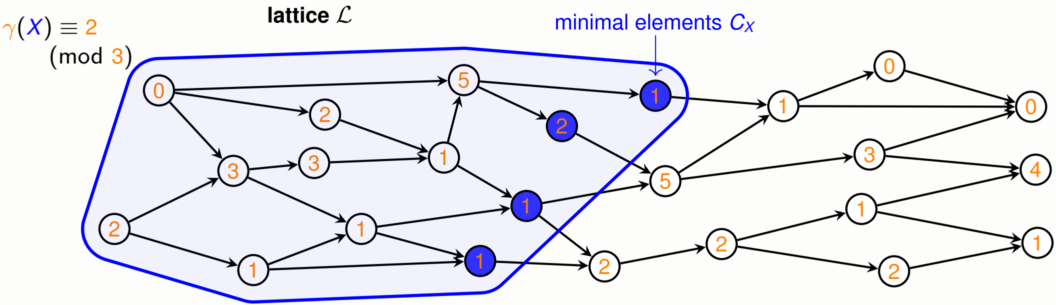

There is a feasible $X \in \mathcal{L}$ with $|C_X| \leq m - 1$.

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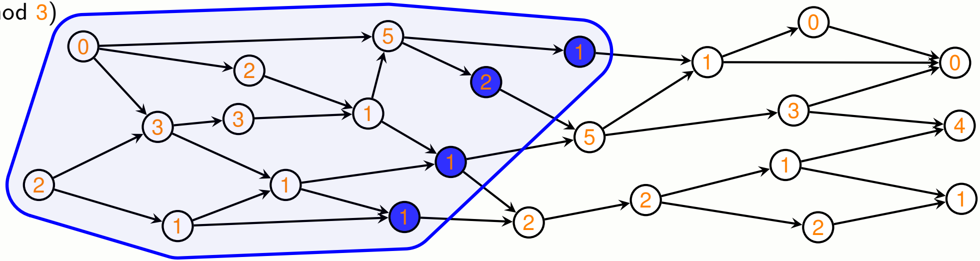

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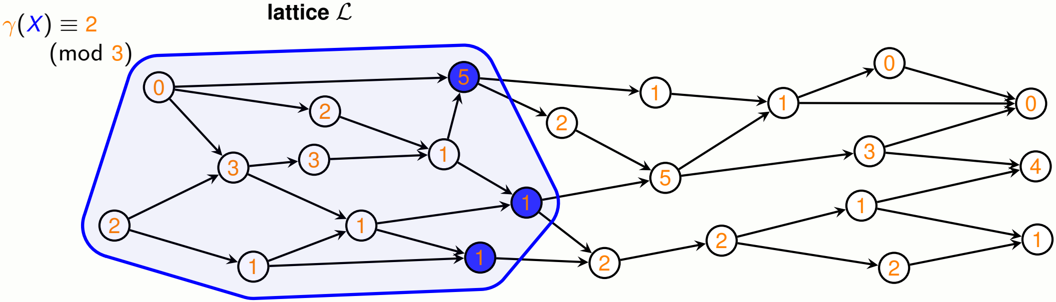

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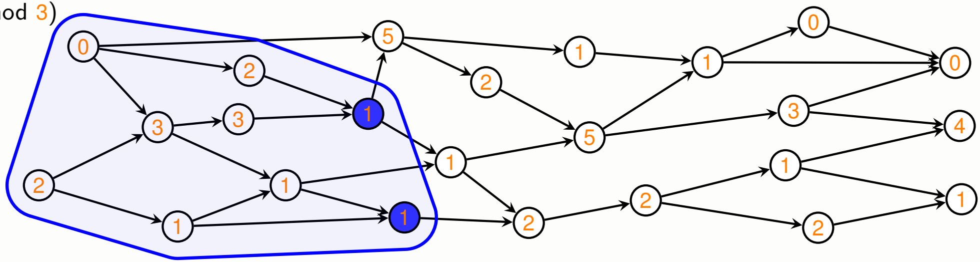

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CCTU with network matrices

"the circulation baseblock"

CCTU with network constraint matrices: Circulations

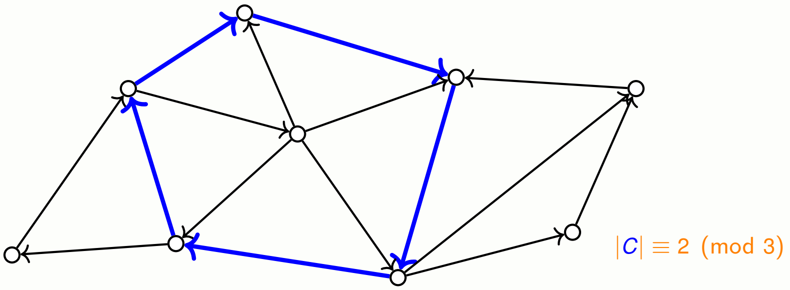
CCTU with transp. netw. matrix

$\min\{\tilde{c}^\top y \colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with network matrix T

⟷

Congruency-constrained circulation

In digraph $G = (V, A)$, find shortest circulation $C \subseteq A$ with $|C| \equiv r \pmod{m}$.



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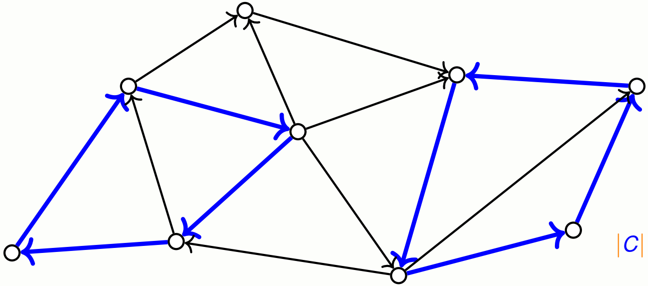

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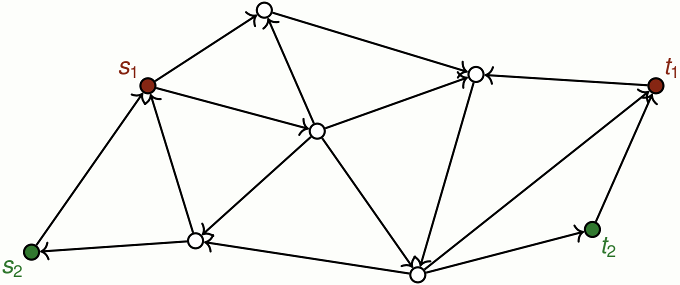

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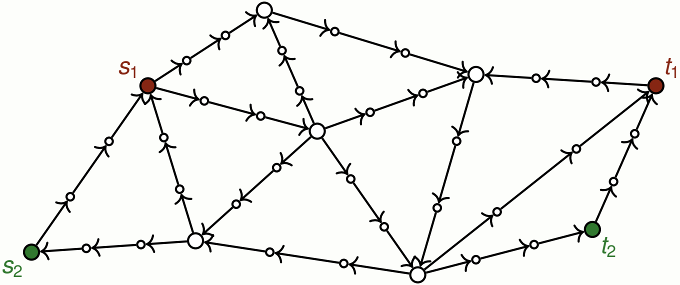

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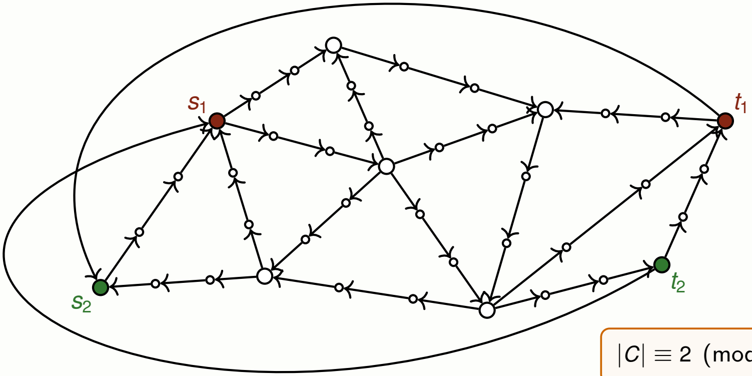


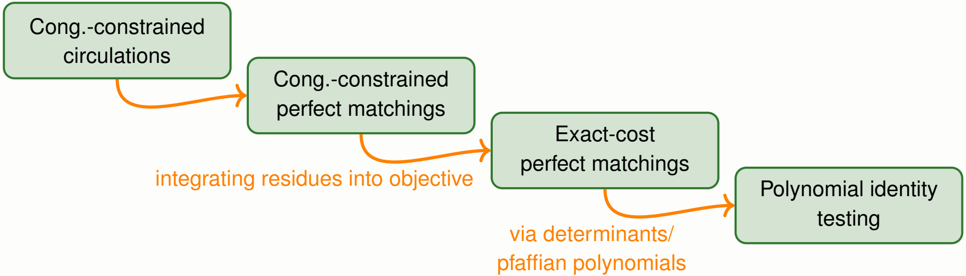
$|C| \equiv 2 \pmod 3 \iff$ paths exist

A reduction

Theorem

∃ strongly poly. randomized alg. for congruency-constrained circulations with unary encoded edge lengths and constant m.

- Our approach:



[Camerini, Galbiati, Maffioli, 1992]

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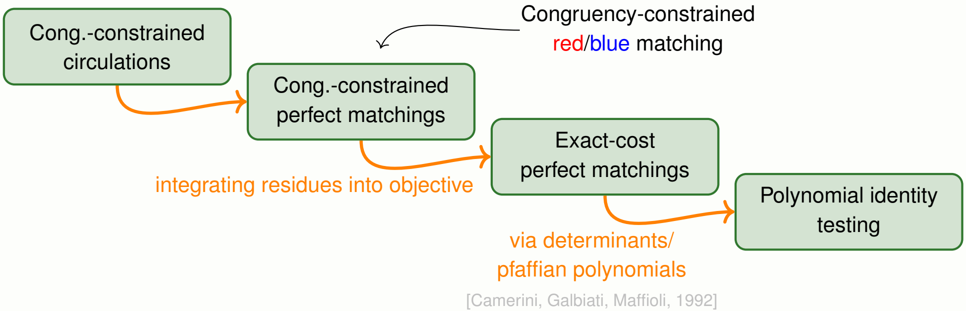
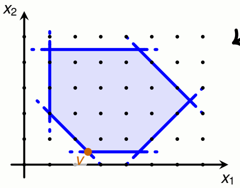


Open questions

Open questions

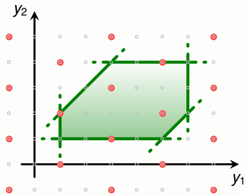


Structural results?

Δ-modular integer programming

$\min\{c^\top x\colon Ax \leqslant b, x \in \mathbb{Z}^n\}$

A is Δ-modular.



Congruency-Constr. TU Prb. (CCTU)

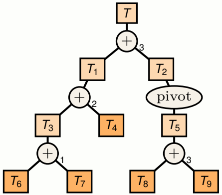
$$\min\left\{\tilde{c}^\top y\colon \begin{array}{c} Ty \leqslant b,\ y \in \mathbb{Z}^n, \\ \gamma^\top y \equiv r \pmod{m} \end{array}\right\}$$

T totally unimodular, modulus m.

?

- What problem should we reduce to?
- How to deal with several differing subdets?

Do we need to go through Seymour's decomposition?

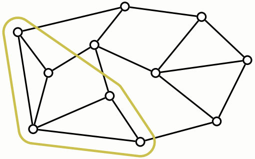


Seymour's TU decomposition

Reduction to base block problems.

- Optimization?
- General m?

- Beyond $m = p^\alpha$ for cut optimization?
- Deterministic approach for circulations?



Base block problems

Interpretation as parity-constrained cut and circulation problems.