A New Contraction Technique with Applications to Congruency-Constrained Cuts

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Introduction: Congruency-Constrained Cuts

Problem Setting, Motivation, and Our Results

Problem Setting

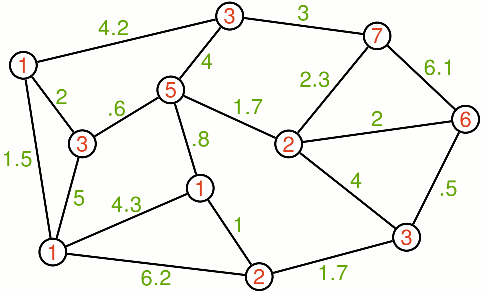
Congruency-Constrained Minimum Cut Problem (CCMC)

Input: Graph $G = (V, E)$, edge weights $w\colon E \to \mathbb{R}_{\geqslant 0}$, vertex multiplicities $\gamma\colon V \to \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{Z}_{>0}$, and $r \in \mathbb{Z}_{\geqslant 0}$.

Goal: Find a minimizer of $\min \left\{ w(\delta(C)) \;\middle|\; \begin{array}{c} \emptyset \subsetneq C \subsetneq V, \\ \sum_{v \in C} \gamma(v) \equiv r \pmod{m} \end{array} \right\}$.

$m = 5$

$r = 2$



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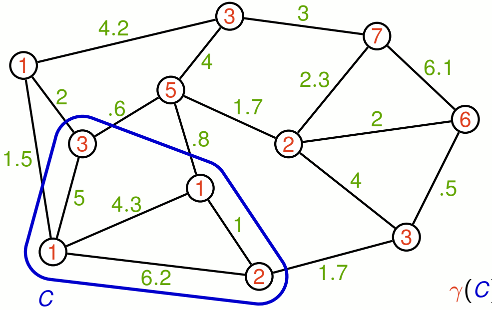

$w(\delta(C)) = 6.6$

$\gamma(C) \equiv 2 \pmod{5}$

Motivation and Prior Results

- Generalization of well-known cut problems:
  - ⇝ Global minimum cuts, minimum s-t-cuts, minimum odd cuts.
- Integer Programming with bounded subdeterminants:

  Can $\min\{c^\top x \mid Ax \leqslant b,\ x \in \mathbb{Z}^n\}$ be solved efficiently if $A \in \mathbb{Z}^{m \times n}$ is m-modular?

  - ⇝ *Bimodular integer programming* ($m = 2$):
    Reduction to parity-constrained cut and flow problems. [Artmann, Weismantel, Zenklusen, 2017]
  - ⇝ CCMC can be reduced to m-modular ILPs.
- Congruency-constrained submodular minimization:
  - ⇝ Efficient algorithm for prime power moduli. [Nägele, Sudakov, Zenklusen, 2018]
  - ⇝ Barriers for composite moduli. [Gopi, 2019]

Our Results

Theorem 1: PRAS for CCMC

CCMC with constant modulus m admits a polynomial time randomized approximation scheme.

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- Approach inspired by Karger's contraction algorithm.
- Novel way of sampling vertex pairs to contract.
  - $\rightsquigarrow$ Using splitting-off techniques from Graph Theory.
- Combination with approximate reduction steps.

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Theorem 2: Exact algorithm for special case

CCMC with modulus $m = pq$ for primes $p \neq q$ admits an exact polynomial time randomized algorithm.

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CCMC with modulus $m = pq$ for primes $p \neq q$ admits an exact polynomial time randomized algorithm.

Theorem 3: Structure for instances with prime moduli

Given a CCMC problem with prime modulus and nonzero optimal value denoted by OPT, there is a randomized algorithm returning polynomially many s-t cut problems such that w.h.p.,

C is solution of (CCMC) problem with value $\leqslant \kappa \cdot \mathrm{OPT}$ $\iff$ C is solution of one of the s-t cut problems with value $\leqslant \kappa \cdot \mathrm{OPT}$.

Karger’s Contraction Algorithm...

. . . and how to adopt it for CCMC.

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Algorithm

while $|V| > 2$ ***do:***

Contract a random edge.

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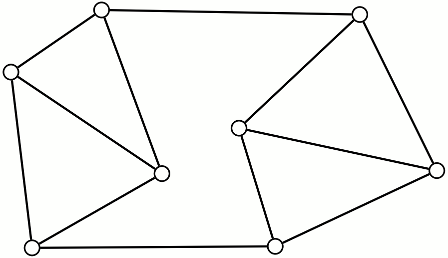

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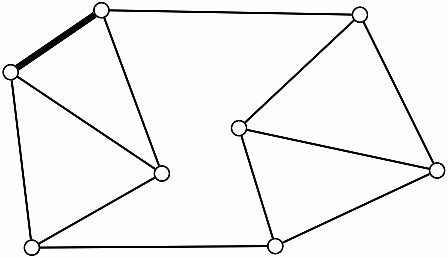

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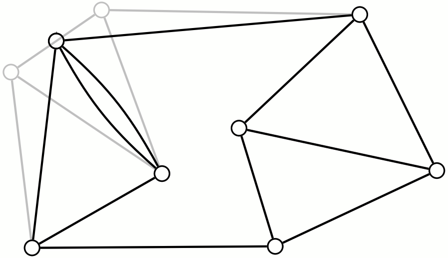

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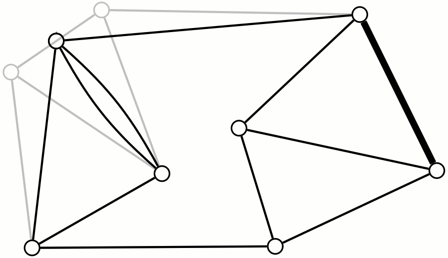

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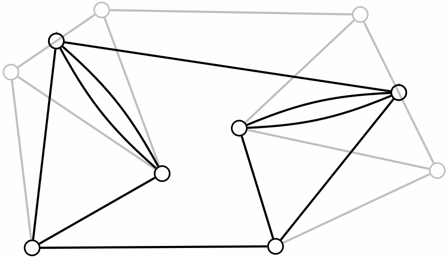

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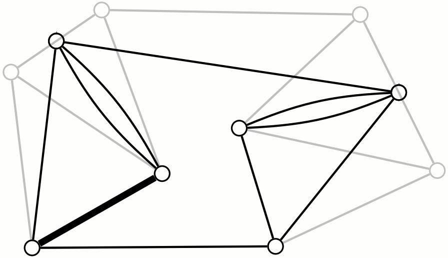

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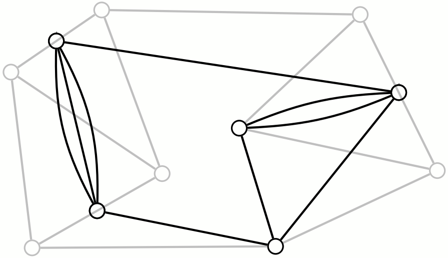

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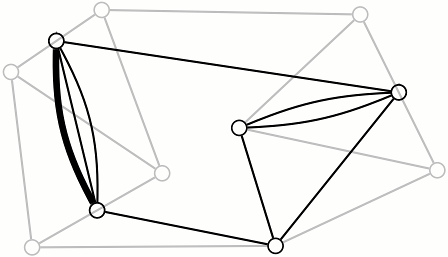

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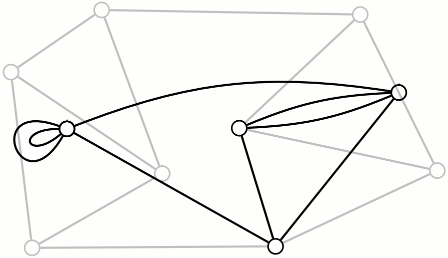

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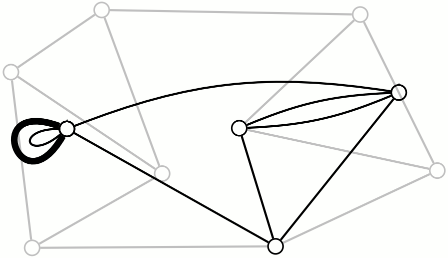

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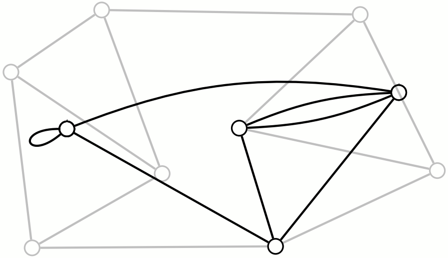

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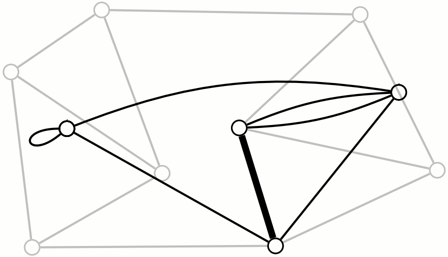

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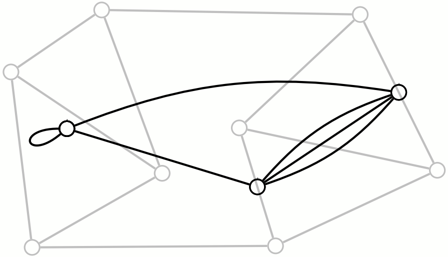

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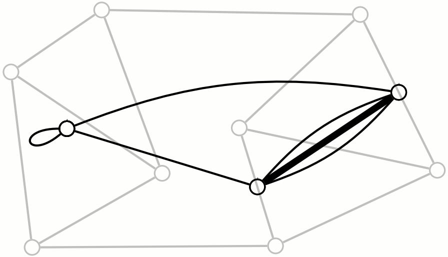

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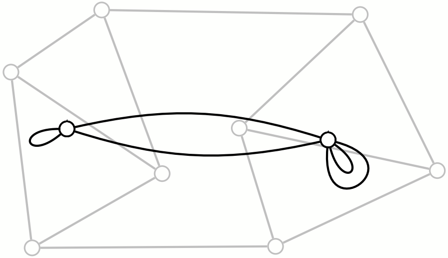

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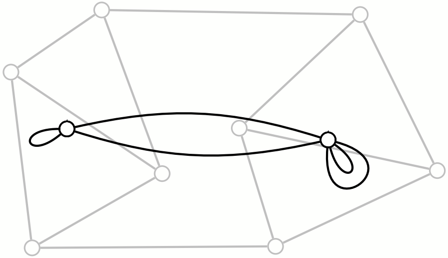

Analysis:

- Singletons are feasible solution candidates.

$$\implies |\delta(v)| \geqslant \mathrm{OPT}\ .$$

- Contractions uniformly at random:

$$\Pr\left[\begin{matrix}\text{contraction is}\\ \text{bad wrt. } C_{\mathrm{OPT}}\end{matrix}\right] = \frac{\mathrm{OPT}}{|E|} = \frac{\mathrm{OPT}}{\frac{1}{2}\sum_{v\in V}|\delta(v)|} \leqslant \frac{2}{|V|}$$

$$\implies \Pr\left[\begin{matrix}\text{no bad}\\ \text{contraction}\end{matrix}\right] \geqslant \prod_{i=3}^{|V|}\left(1-\frac{2}{i}\right) = \Omega\left(\frac{1}{|V|^2}\right).$$

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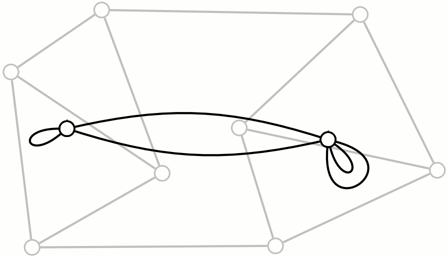

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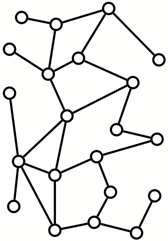
If $\sum_{v\in V}|\delta(v)| \geqslant \varepsilon\cdot|V|\cdot\mathrm{OPT}$, Karger until ${}^2/_\varepsilon$ vertices remain succeeds with probability $\Omega(|V|^{-{}^2/_\varepsilon})$.

⇝ Enumerate remaining options.

What fails with $\gamma(C) \equiv r \pmod{m}$?

$G = (V, E)$



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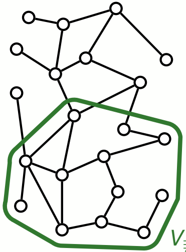

$V_{\not\equiv 0} = \{v \in V \mid \gamma(v) \not\equiv 0 \pmod{m}\}$

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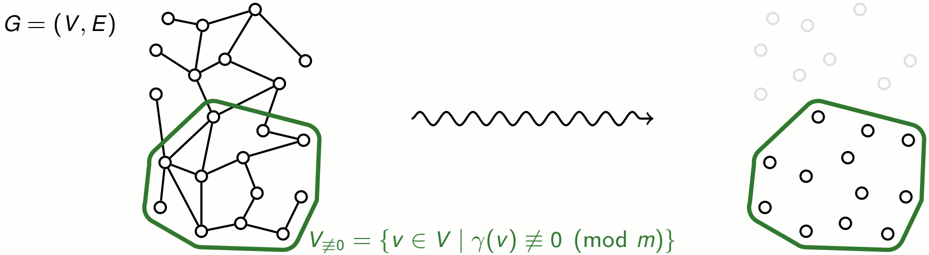

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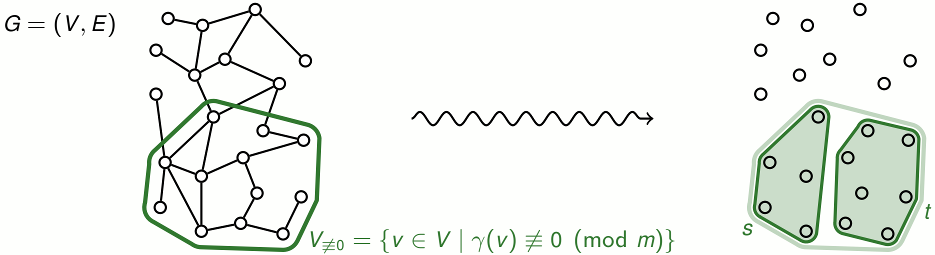

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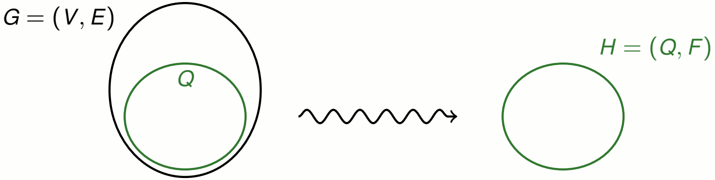
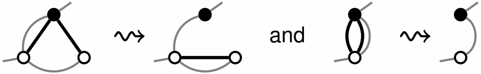
- Fundamental technique from Graph Theory [Lovász, 1976 & 1979] [Mader, 1978]



- Two operations: 

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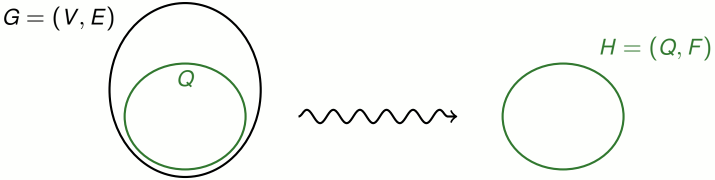
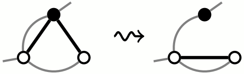
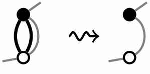


- Two operations:  and 

Theorem [Lov. '76]

Let G Eulerian, then edges can be split from $v \in V \setminus Q$ in pairs such that

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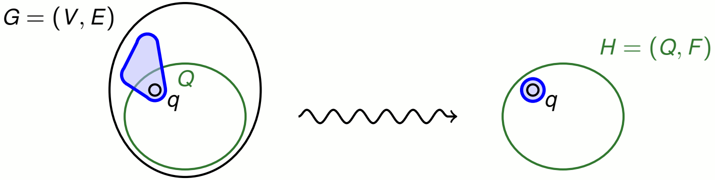
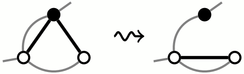
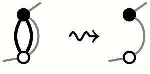


- Two operations:  and 

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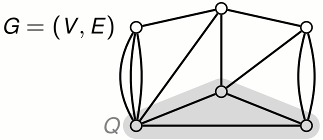

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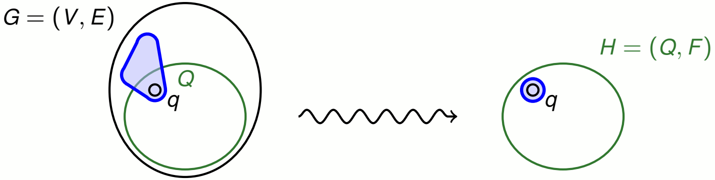
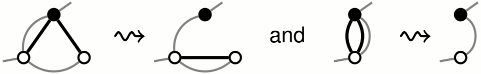


- Two operations: 

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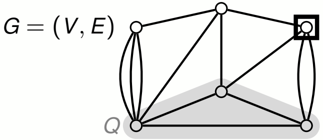

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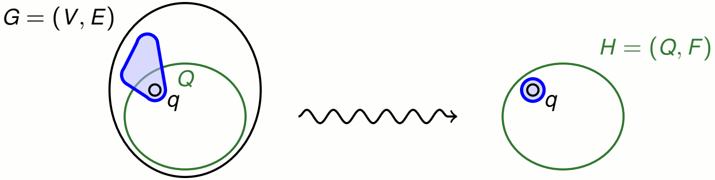

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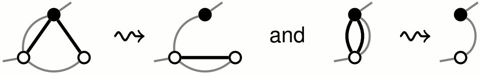

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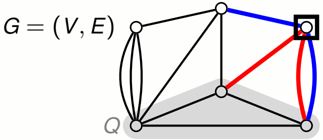

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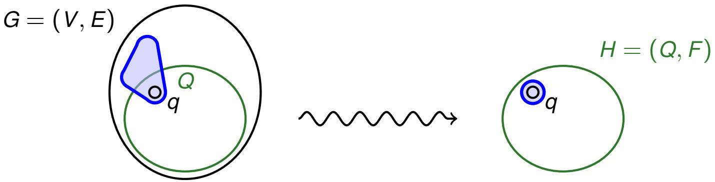
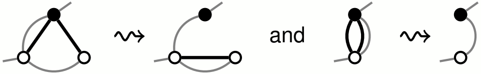


- Two operations: 

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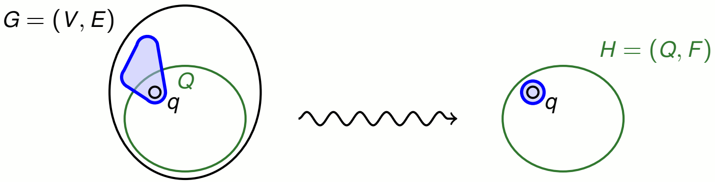
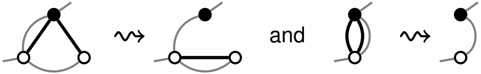
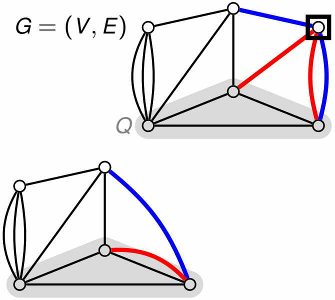


- Two operations: 



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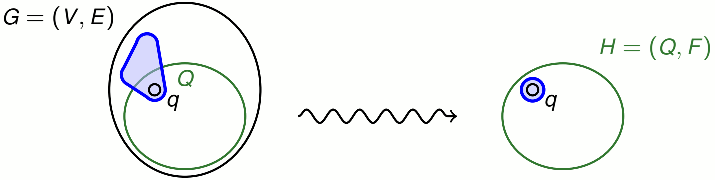
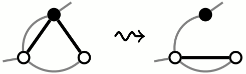
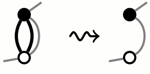
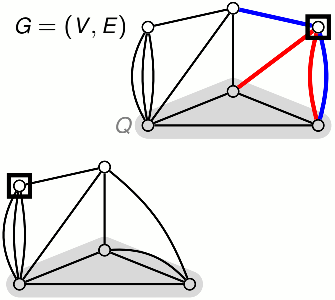


- Two operations:  and 



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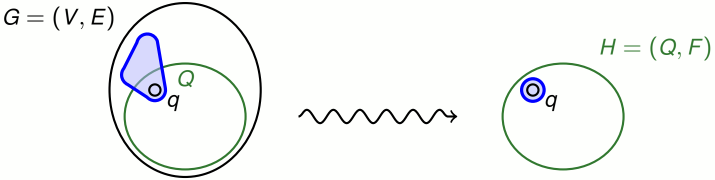
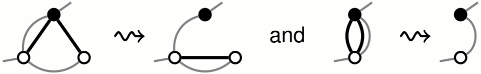
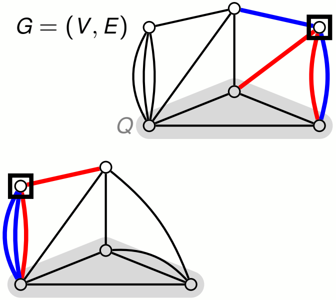


- Two operations: 



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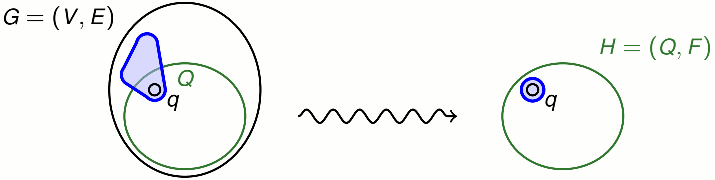
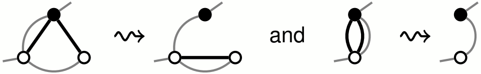
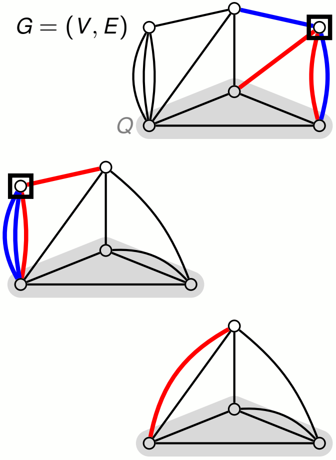


- Two operations: 



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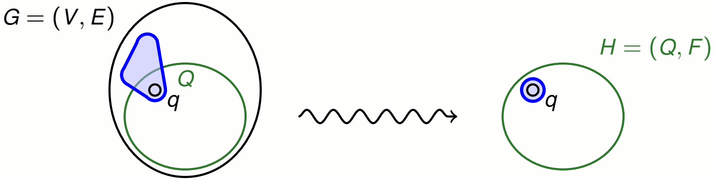
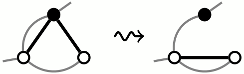
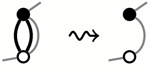
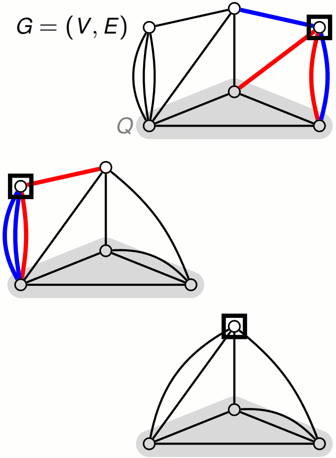


- Two operations:  and 



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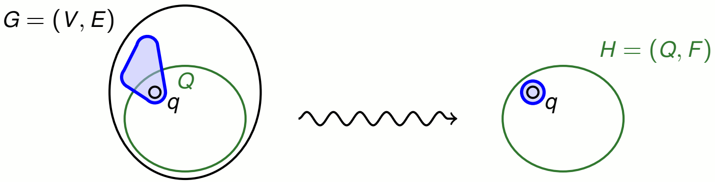
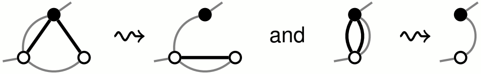
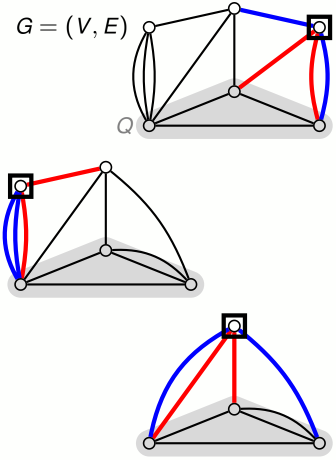


- Two operations: 



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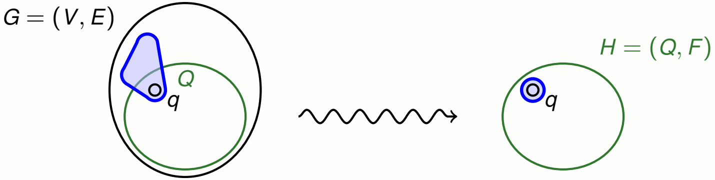
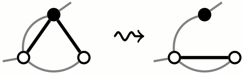
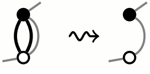
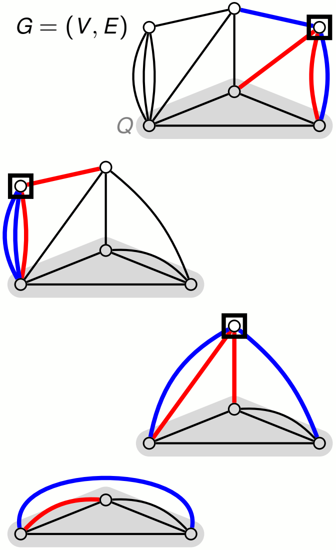


- Two operations:  and 



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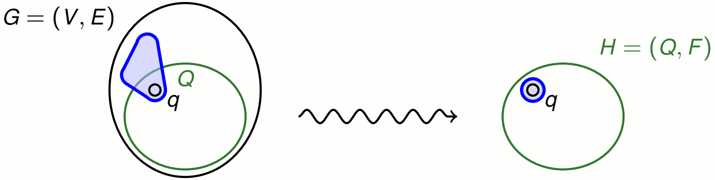

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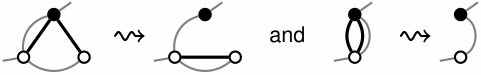

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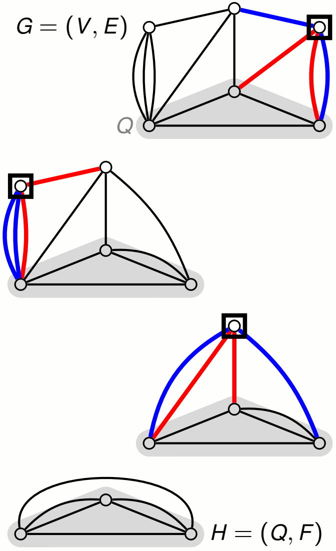

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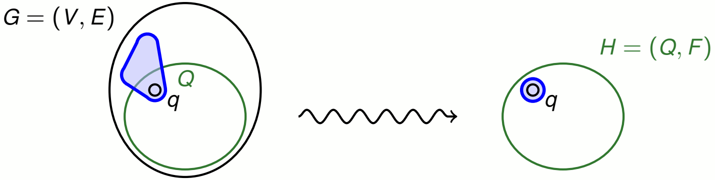
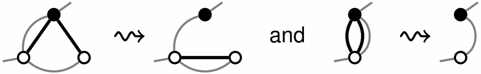
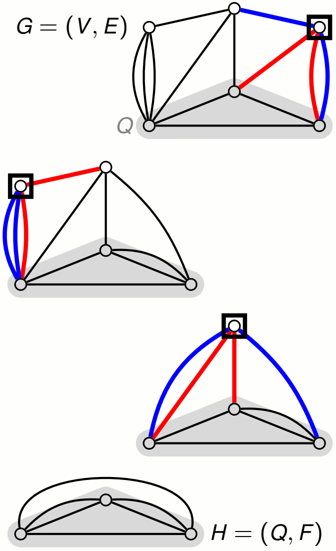


- Two operations: 



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- Weighted algorithmic version: Combination with ideas of Frank. [Frank, 1992]

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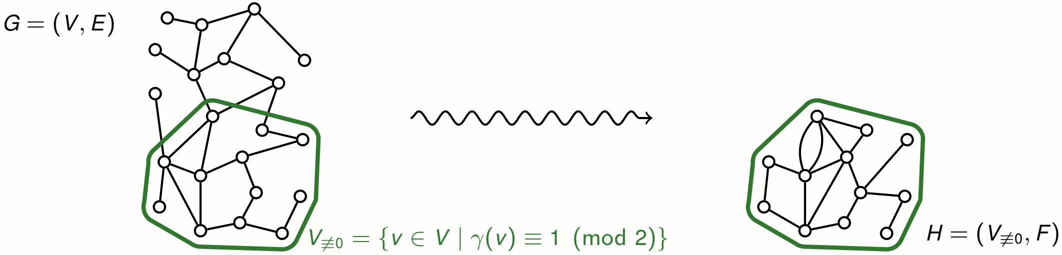
- CCMC with $m = 2$ and $r = 1$, i.e., constraint $\gamma(C) \equiv 1 \pmod 2$.



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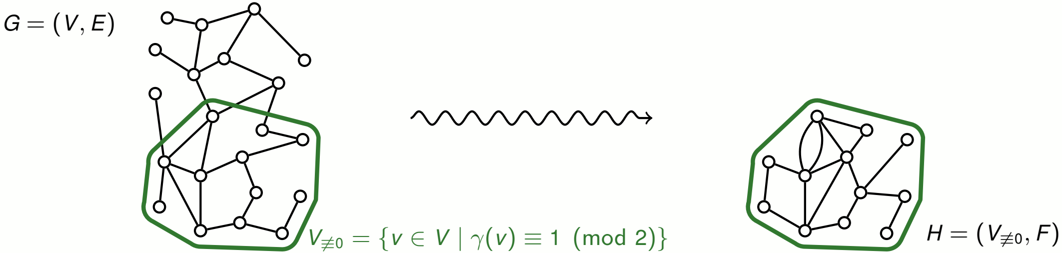

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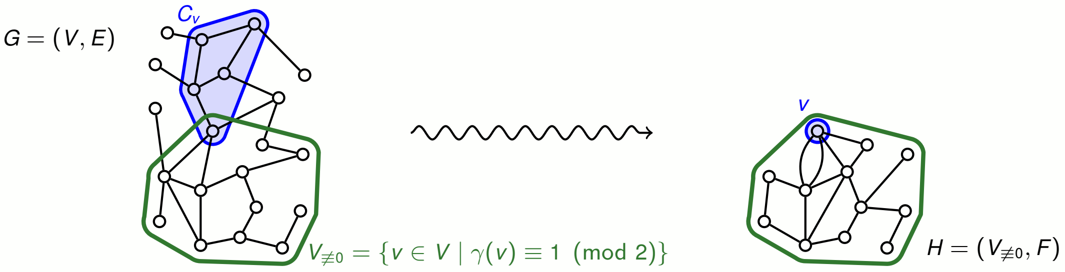

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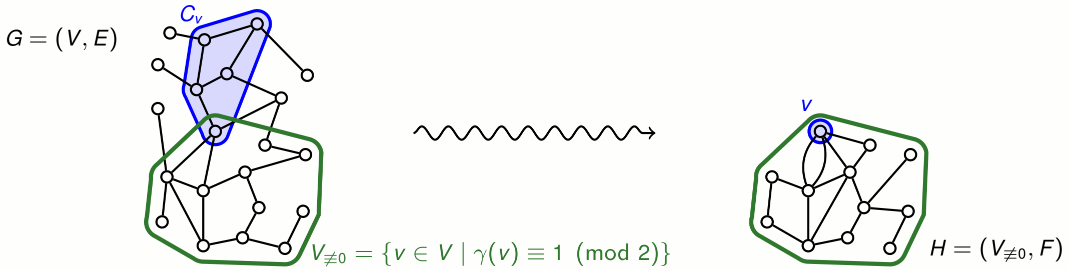

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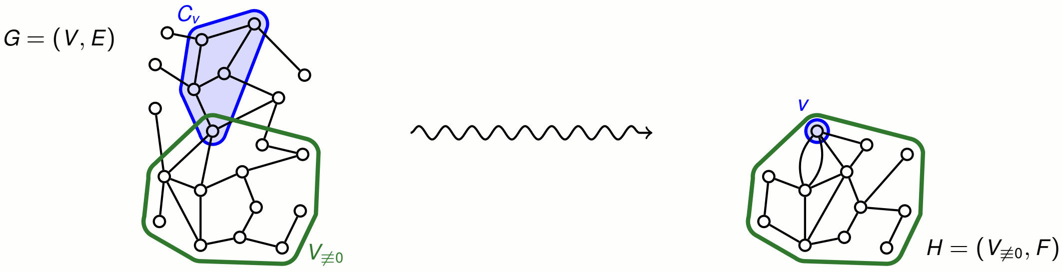

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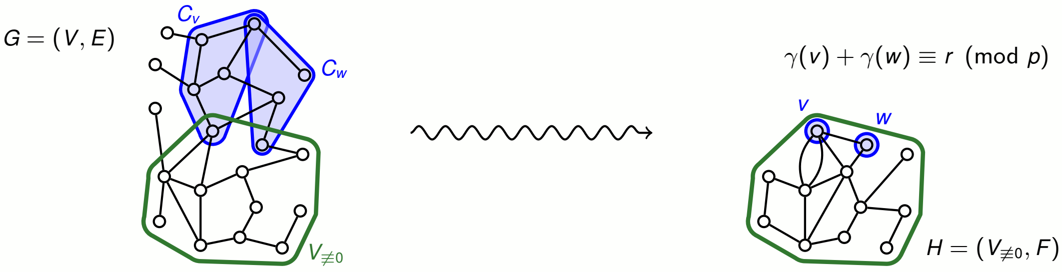

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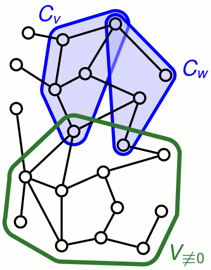

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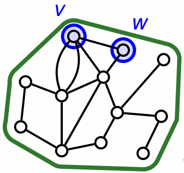

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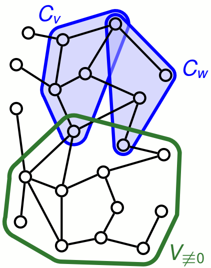

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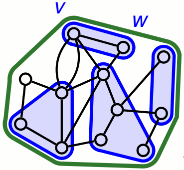

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Thm. **(Cauchy-Davenport)** Among any p nonzero elements of $\mathbb{Z}/p\mathbb{Z}$, there is a subset summing to $r \pmod p$.

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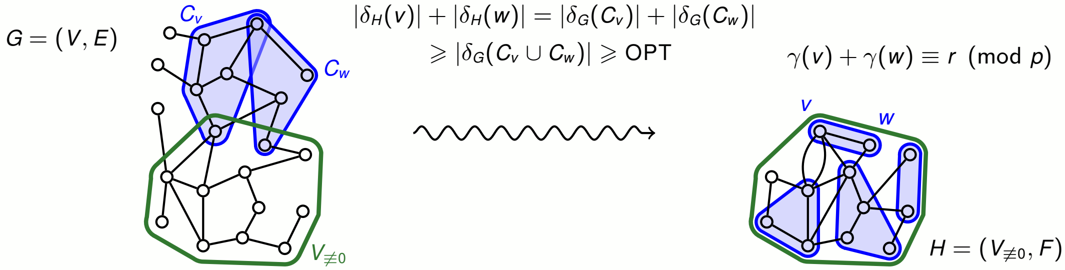

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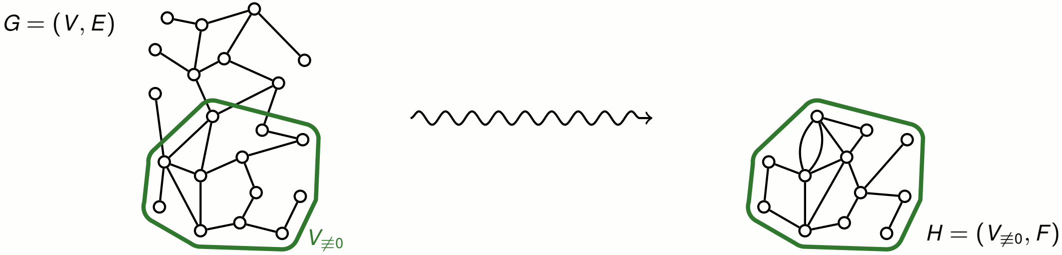

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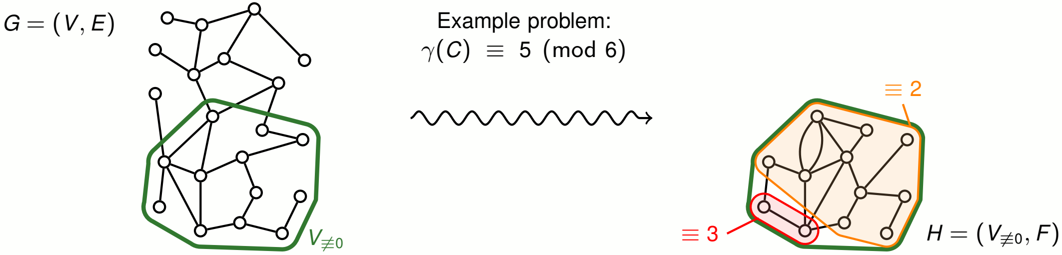

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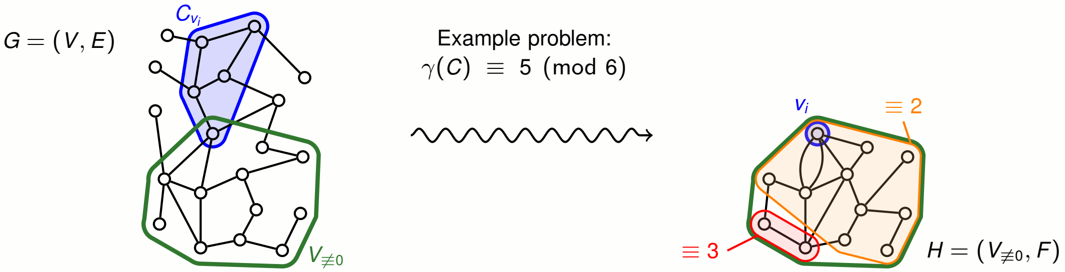

- There is $q \in [m-1]$ and many vertices $v_i \in V_{\neq 0}$ with

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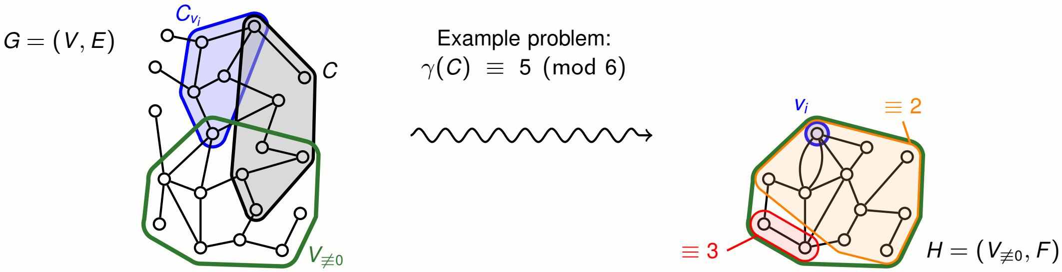

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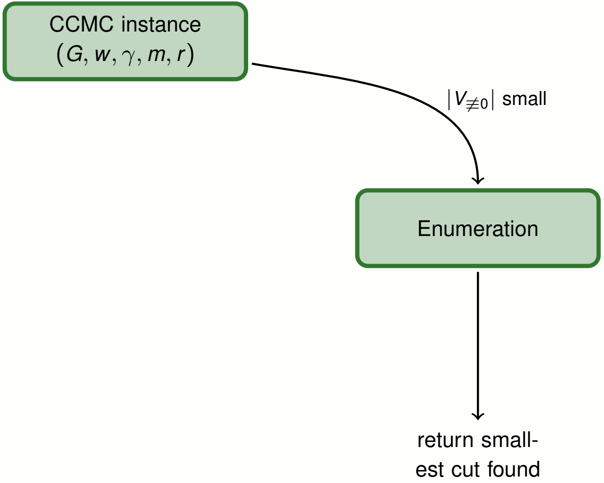
$\Longrightarrow$ Cheap residue correction by multiples of q—leaves problem modulo $\gcd(m, q)$.

The Complete Algorithm

CCMC instance
(G, w, γ, m, r)

The Complete Algorithm



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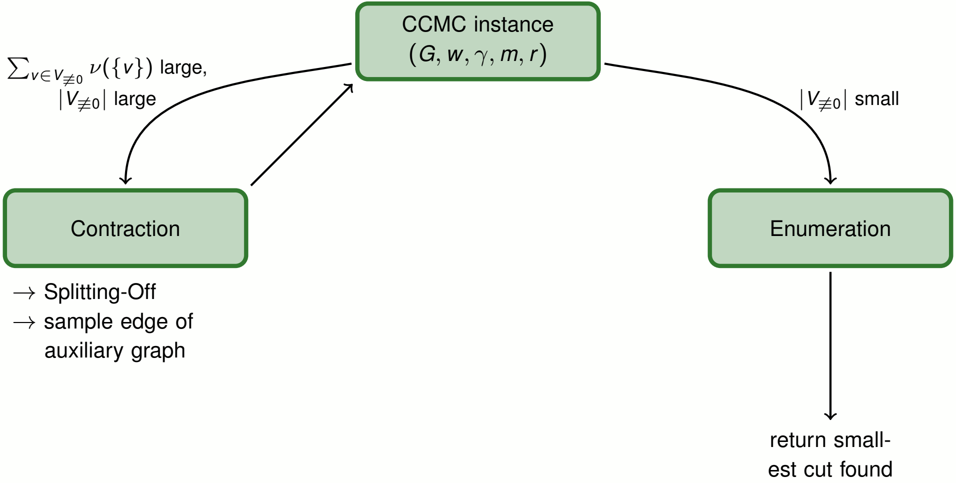

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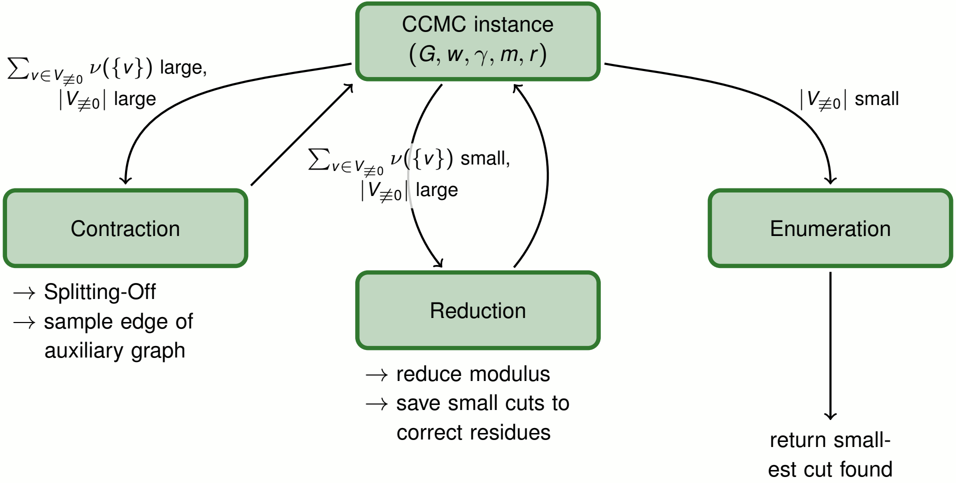

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