Congruency-Constrained TU Problems

Beyond the Bimodular Case

Martin Nägele Richard Santiago Rico Zenklusen

Institute for Operations Research
ETH Zürich

The agenda for today

1. **Motivation & background**
   bounded subdeterminant IPs — successes in the bimodular case — new results

2. **Structural aspects of CCTU problems and their solutions**
   a decomposition lemma — proximity — flatness

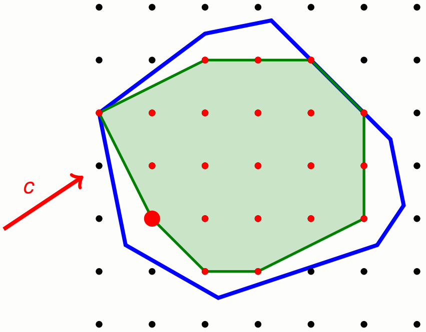
3. **A decomposition approach to CCTU problems**
   Seymour's decomposition — deciding feasibility for modulus 3

4. **Base block problems**
   congruency-constrained minimum cuts and circulations

Motivation & Background

bounded subdeterminant IPs — successes
in the bimodular case — new results

Towards general classes of efficiently solvable IPs

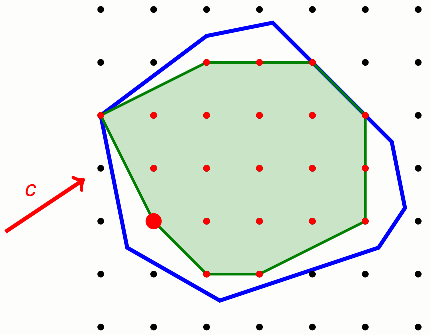


Integer Linear Programming (IP)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve

$$\min\{c^\top x \colon Ax \leqslant b,\, x \in \mathbb{Z}^n\} \ .$$

Towards general classes of efficiently solvable IPs



Integer Linear Programming (IP)

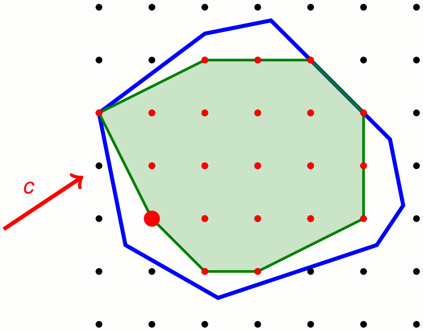
Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve

$$\min\{c^\top x \colon Ax \leqslant b,\ x \in \mathbb{Z}^n\} \ .$$

Two classes of efficiently solvable IPs

- If $n = O(1)$ or $m = O(1)$:
  $\rightarrow$ Lenstra's Algorithm [Lenstra 1983].
- If A is totally unimodular (TU):
  $\rightarrow$ Integral relaxation.

Towards general classes of efficiently solvable IPs



Integer Linear Programming (IP)

Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$, solve

$$\min\{c^\top x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}\ .$$

Two classes of efficiently solvable IPs

- If $n = O(1)$ or $m = O(1)$:
  $\rightarrow$ Lenstra's Algorithm [Lenstra 1983].
- If A is totally unimodular (TU):
  $\rightarrow$ Integral relaxation.

What if minors, in absolute value, are still bounded, but not by 1?

Bounded subdeterminants

Δ-modular Integer Programming

Given a constant $\Delta \in \mathbb{Z}_{>0}$, can integer linear programs

$$\min\{c^\top x \colon Ax \leq b,\ x \in \mathbb{Z}^n\}$$

with Δ-modular constraint matrix A be solved efficiently?

- $A \in \mathbb{Z}^{m \times n}$ is Δ-modular if
  - → $\mathrm{rank}(A) = n$, and
  - → absolute values of $n \times n$ subdeterminants are bounded by Δ
- Δ-modularity is more general than *total* Δ-modularity

Bounded subdeterminants

Δ-modular Integer Programming

Given a constant $\Delta \in \mathbb{Z}_{>0}$, can integer linear programs

$$\min\{c^\top x\colon Ax \le b,\ x \in \mathbb{Z}^n\}$$

with Δ-modular constraint matrix A be solved efficiently?

- $A \in \mathbb{Z}^{m \times n}$ is Δ-modular if
  - → $\text{rank}(A) = n$, and
  - → absolute values of $n \times n$ subdeterminants are bounded by Δ
- Δ-modularity is more general than *total* Δ-modularity

Known results

✓ $\Delta = 1$: easy

✓ $\Delta = 2$: Bimodular Integer Programming (BIP)

[Artmann, Weismantel, and Zenklusen, STOC 2017]

✓ Arbitrary constant Δ, at most 2 non-zeros per row

[Fiorini, Joret, Weltge, and Yuditsky, FOCS 2021]

Bimodular integer programs

Bimodular integer programming (BIP)

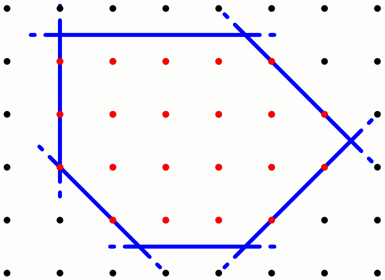
Given $A \in \mathbb{Z}^{k\times n}$, $b \in \mathbb{Z}^m$, and $c \in \mathbb{Z}^n$ such that A has full column rank and all $n \times n$ minors in $\{-2,-1,0,1,2\}$, solve

$$\min\{c^\top x \colon Ax \leqslant b,\ x \in \mathbb{Z}^n\} \ .$$

Theorem

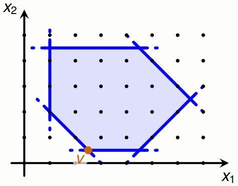
BIP can be solved in strongly polynomial time.

[Artmann, Weismantel, and Zenklusen, STOC 2017]



$$\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 1 & 1 \\ 0 & 2 \\ -1 & 0 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 4 \\ 9 \\ 9 \\ -1 \\ -3 \end{pmatrix}$$

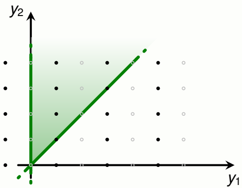
The approach



Bimodular Integer Program (BIP)

$$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$$

s.t. A bimodular, relaxation fractional.



Conic Parity TU Problem (CPTU)

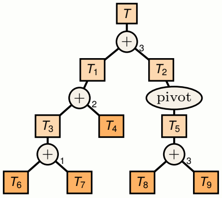
$$\min\{\tilde{c}^\top y \colon Ty \leqslant 0, y \in \mathbb{Z}^n, y(S) \text{ odd}\}$$

with T totally unimodular, and $S \subseteq [n]$.

Theorem

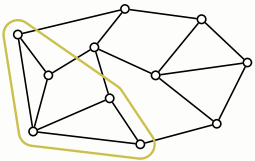
BIP can be solved in strongly polynomial time.

[Artmann, Weismantel, and Zenklusen, STOC 2017]



Seymour's TU decomposition

Exploited for reduction to parity-constrained base block problems.



Base block problems

Interpretation as parity-constrained cut and circulation problems

CCTU problems

Congruency-constrained TU problems (CCTU)

Let $T \in \{-1,0,1\}^{k\times n}$ totally unimodular, $b \in \mathbb{Z}^k$, $\gamma \in \mathbb{Z}^n$, $m \in \mathbb{Z}_{>0}$, and $r \in \mathbb{Z}$. Solve

$$\min\left\{c^\top x\colon\, Tx \le b,\ \gamma^\top x \equiv r \pmod{m},\ x \in \mathbb{Z}^n\right\}\ .$$

- Special case of m-modular IP

Our results

Congruency-constrained TU problems (CCTU)

Let $T \in \{-1,0,1\}^{k\times n}$ totally unimodular, $b \in \mathbb{Z}^k$, $\gamma \in \mathbb{Z}^n$, $m \in \mathbb{Z}_{>0}$, and $r \in \mathbb{Z}$. Solve

$$\min\left\{c^\top x \colon Tx \le b,\ \gamma^\top x \equiv r \pmod{m},\ x \in \mathbb{Z}^n\right\} .$$

Theorem 1: Feasibility for $m = 3$

$\exists$ strongly poly. randomized alg. for CCTU feasibility with $m = 3$.

Our results

Congruency-constrained TU problems (CCTU)

Let $T \in \{-1,0,1\}^{k\times n}$ totally unimodular, $b \in \mathbb{Z}^k$, $\gamma \in \mathbb{Z}^n$, $m \in \mathbb{Z}_{>0}$, and $r \in \mathbb{Z}$. Solve

$$\min\left\{c^\top x \colon Tx \le b,\ \gamma^\top x \equiv r \pmod{m},\ x \in \mathbb{Z}^n\right\} .$$

Theorem 1: Feasibility for $m = 3$

$\exists$ strongly poly. randomized alg. for CCTU feasibility with $m = 3$.

Theorem 2: Flat or feasible

Either $\exists$ flat constraint of width at most $m-2$, or a feasible CCTU solution can be found in strongly poly. time.

Theorem 3: Proximity

If feasible, then for any x_0 optimal for a CCTU relaxation, $\exists\, x^*$ optimal for the problem with $\|x^* - x_0\|_\infty \le m-1$.

Structural results on CCTU problems

decomposition lemma — flatness — proximity

A decomposition lemma for solutions of TU systems

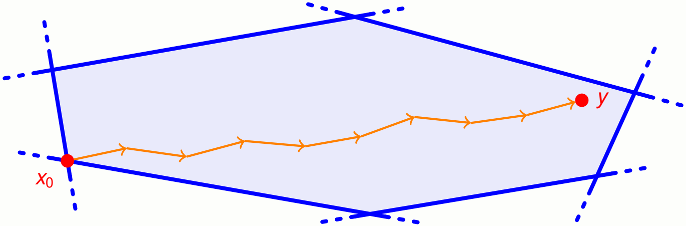
Decomposition lemma

Let $x_0, y \in \mathbb{Z}^n$ be solutions of a TU system $Tx \leq b$. There are $y^i \in \mathbb{Z}^n$ with

$$y = x_0 + \sum_{i=1}^{\ell} y^i \quad , \quad \text{and}$$

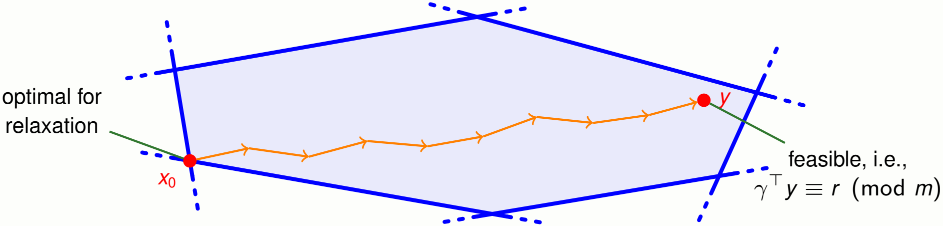
(i) $|d^\top y^i| \leq 1$ for all d that are TU-appendable to T, and

(ii) $\forall S \subseteq [\ell]\colon\ x_0 + \sum_{i \in S} y^i$ is feasible for $Tx \leq b$.



Proximity

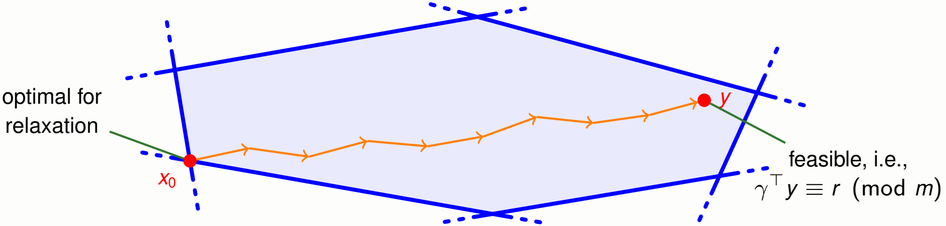
$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$



Proximity

$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$

$$\implies \qquad \gamma^\top y \equiv \gamma^\top x_0 + \gamma^\top y^1 + \gamma^\top y^2 + \gamma^\top y^3 + \gamma^\top y^4 + \gamma^\top y^5 + \gamma^\top y^6 + \ldots + \gamma^\top y^\ell \equiv r \pmod{m}$$



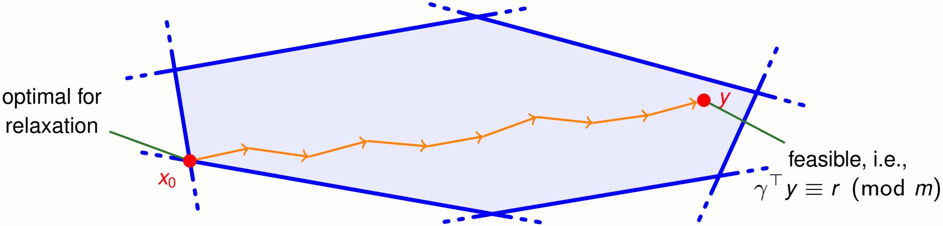
Proximity

Lemma

For any m integers, there is a subset with sum $\equiv 0 \pmod{m}$.

$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$

$$\implies \quad \gamma^\top y \equiv \gamma^\top x_0 + \gamma^\top y^1 + \gamma^\top y^2 + \gamma^\top y^3 + \gamma^\top y^4 + \gamma^\top y^5 + \gamma^\top y^6 + \ldots + \gamma^\top y^\ell \equiv r \pmod{m}$$



Proximity

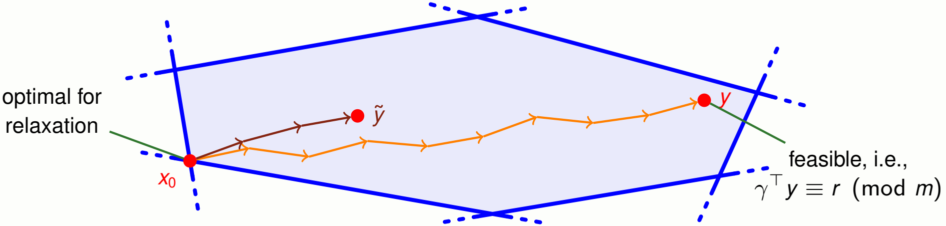
Lemma

For any m integers, there is a subset with sum $\equiv 0 \pmod m$.

$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$

$$\Longrightarrow \qquad \gamma^\top y \equiv \gamma^\top x_0 + \gamma^\top y^1 + \gamma^\top y^2 + \gamma^\top y^3 + \gamma^\top y^4 + \gamma^\top y^5 + \gamma^\top y^6 + \ldots + \gamma^\top y^\ell \equiv r \pmod m$$

$\exists\, S \subseteq [\ell]$ with $|S| \leq m-1$
s.th. $\tilde{y} := x_0 + \sum_{i \in S} y^i$ is feasible.



Proximity

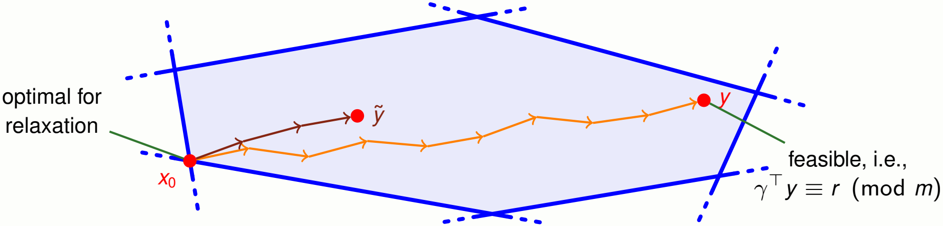
Lemma

For any m integers, there is a subset with sum $\equiv 0 \pmod{m}$.

$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$

$$\implies \quad \gamma^\top y \equiv \gamma^\top x_0 + \gamma^\top y^1 + \gamma^\top y^2 + \gamma^\top y^3 + \gamma^\top y^4 + \gamma^\top y^5 + \gamma^\top y^6 + \ldots + \gamma^\top y^\ell \equiv r \pmod{m}$$

$\exists\, S \subseteq [\ell]$ with $|S| \leq m-1$ s.th. $\tilde{y} := x_0 + \sum_{i \in S} y^i$ is feasible. $\longrightarrow$ $|d^\top(\tilde{y} - x_0)| \leq m-1$ for all TU-appendable d



Proximity

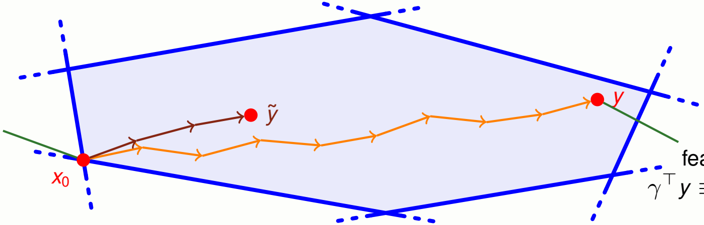
Lemma

For any m integers, there is a subset with sum $\equiv 0 \pmod{m}$.

$$y = x_0 + y^1 + y^2 + y^3 + y^4 + y^5 + y^6 + \ldots + y^\ell$$

$$\Longrightarrow \quad \gamma^\top y \equiv \gamma^\top x_0 + \gamma^\top y^1 + \gamma^\top y^2 + \gamma^\top y^3 + \gamma^\top y^4 + \gamma^\top y^5 + \gamma^\top y^6 + \ldots + \gamma^\top y^\ell \equiv r \pmod{m}$$

$\exists\, S \subseteq [\ell]$ with $|S| \leq m-1$ s.th. $\tilde{y} := x_0 + \sum_{i \in S} y^i$ is feasible. → $|d^\top(\tilde{y} - x_0)| \leq m-1$ for all TU-appendable d → $\|\tilde{y} - x_0\|_\infty \leq m-1$

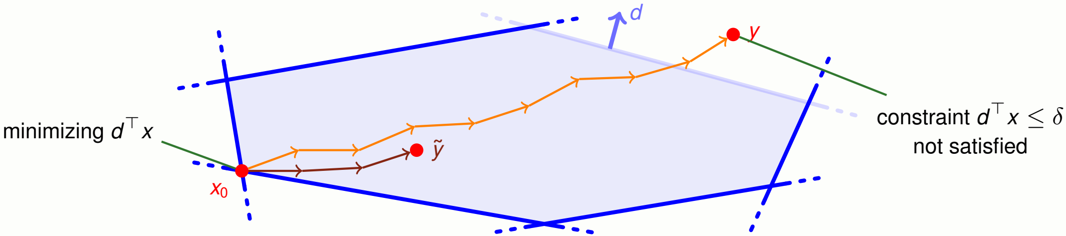


Flatness or feasibility

A constraint $d^\top x \leq \delta$ is redundant if the width in direction d is at least $m - 1$.

$\Longrightarrow$

Either some constraint widht is at most $m - 2$, or the problem is feasible.



A decomposition approach to CCTU problems

deciding feasibility of CCTU problems with $m = 3$

Decomposition of TU matrices

Theorem: Seymour's decomposition [Seymour, 1980]

For every TU matrix $T \in \mathbb{Z}^{k \times n}$, one of the following cases holds:

(i) T or $T^\top$ is a network matrix.

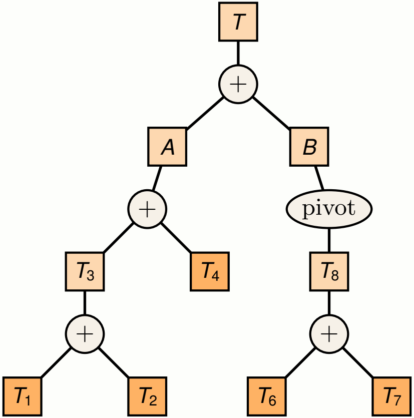
(ii) T is, after repeatedly deleting unit or duplicate rows/columns, changing the sign of a row/column, and row/column permutations equal to one of

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(iii) T is, possibly after row/column permutations and pivoting once, of the form

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix},$$

where A and B each have at least 2 columns.



General idea for CCTU:

Reduce to smaller subproblems along decomposition, solve base blocks directly.

Applying the decomposition

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv r \pmod{m}$$

Applying the decomposition

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$
$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv r \pmod{m}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{m}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv r \pmod{m}$.

B-subproblem

$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{m}$$

Applying the decomposition

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv r \pmod{m}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{m}$$

B-subproblem

$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{m}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv r \pmod{m}$.

Decomposition lemma: If feasible, there is a solution with α, β in intervals of length $m - 1$.

Applying the decomposition

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv r \pmod{m}$$

A-subproblem

$$A x_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{m}$$

B-subproblem

$$B x_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{m}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv r \pmod{m}$.

Decomposition lemma: If feasible, there is a solution with α, β in intervals of length $m - 1$.

Natural strategy: Recurse on constantly many subproblems, check for “compatible” solutions.

Subproblem patterns (for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

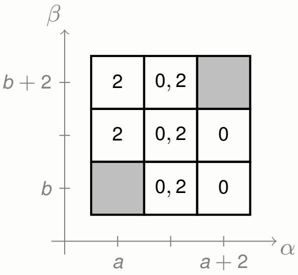
A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

B-subproblem

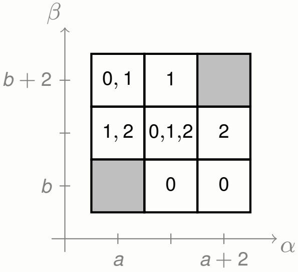
$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.



Decomposition lemma: If feasible, there is a solution with α, β in intervals of length 2.

Natural strategy: Recurse on constantly many subproblems, check for “compatible” solutions.



Subproblem patterns (for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$
$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

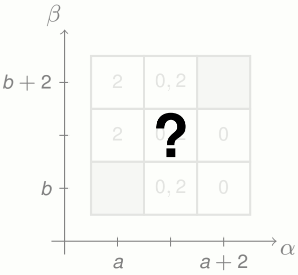
A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

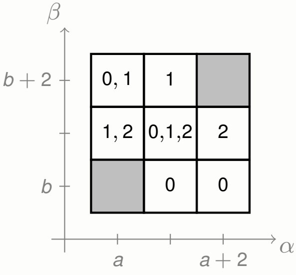
B-subproblem

$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$



Issue: Recursing is efficient only for log-depth decomposition trees.

- Can only completely determine the pattern of the smaller subproblem!



Studying patterns I

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$

$$h^\top x_A = \beta$$

$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

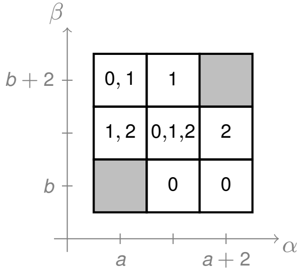
For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

B-subproblem

$$B x_B \leq b_B - \beta g$$

$$f^\top x_B = \alpha$$

$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$



Studying patterns I

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$

$$h^\top x_A = \beta$$

$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

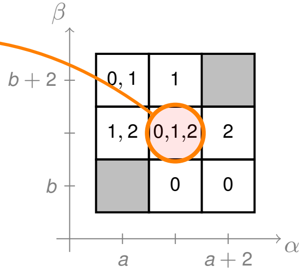
B-subproblem

$$B x_B \leq b_B - \beta g$$

$$f^\top x_B = \alpha$$

$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Any solution x_A of the A-subproblem for $(\alpha, \beta) = (a+1, b+1)$ can be complemented to a solution (x_A, x_B) with residue 2. ✓



Studying patterns II

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$

$$h^\top x_A = \beta$$

$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

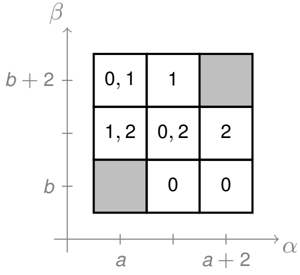
B-subproblem

$$B x_B \leq b_B - \beta g$$

$$f^\top x_B = \alpha$$

$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.



Studying patterns II

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$Ax_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

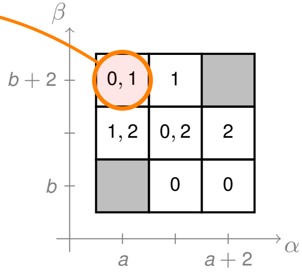
For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

B-subproblem

$$Bx_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Want a solution with residue 1 or 2, i.e., of

$$Ax_A \le b_A - a \cdot e$$
$$h^\top x_A = b + 2$$
$$\gamma_A^\top x_A \in \{1, 2\} \pmod{3}$$



Studying patterns II

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

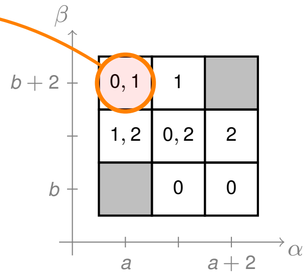
B-subproblem

$$B x_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Want a solution with residue 1 or 2, i.e., of

$$A x_A \le b_A - a \cdot e$$
$$h^\top x_A = b + 2$$
$$\gamma_A^\top x_A \in \{1, 2\} \pmod{3}$$

Easier problem! ✓



Studying patterns II

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod 3$$

***A*-subproblem**

$$A x_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod 3$$

***B*-subproblem**

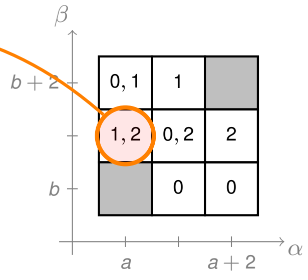
$$B x_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod 3$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod 3$.

Want a solution with residue 0 or 1, i.e., of

$$A x_A \le b_A - a \cdot e$$
$$h^\top x_A = b + 2$$
$$\gamma_A^\top x_A \in \{0, 1\} \pmod 3$$

Easier problem! ✓



Studying patterns II

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

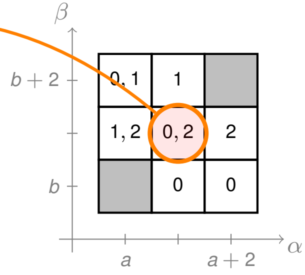
B-subproblem

$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Want a solution with residue 0 or 2, i.e., of

$$A x_A \leq b_A - a \cdot e$$
$$h^\top x_A = b + 2$$
$$\gamma_A^\top x_A \in \{0, 2\} \pmod{3}$$

Easier problem! ✓



Studying patterns III

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

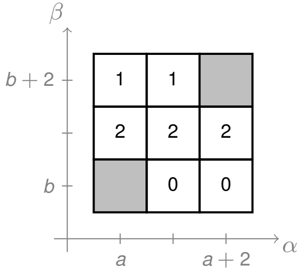
A-subproblem

$$A x_A \leq b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

B-subproblem

$$B x_B \leq b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$



Studying patterns III

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$

$$h^\top x_A = \beta$$

$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

B-subproblem

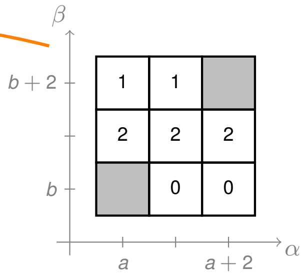
$$B x_B \leq b_B - \beta g$$

$$f^\top x_B = \alpha$$

$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

Linear *pattern:* One feasible residue per (α, β)-pair $\implies$ residue is linear in α and β.



Studying patterns III

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

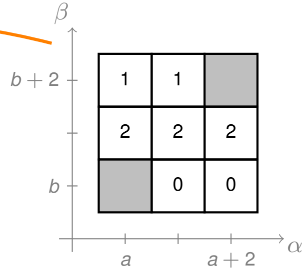
B-subproblem

$$B x_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Linear *pattern:* One feasible residue per (α, β)-pair
$\implies$ residue is linear in α and β.

Reduction to

$$A x_A + \alpha e \le b_A$$
$$h^\top x_A - \beta = 0$$
$$(\alpha, \beta) \in \Pi$$
$$\gamma_A^\top x_A + r_1 \alpha + r_2 \beta + r_0 \equiv 2 \pmod{3}$$



Studying patterns III

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \le \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$Ax_A \le b_A - \alpha e$$
$$h^\top x_A = \beta$$
$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$,
we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

B-subproblem

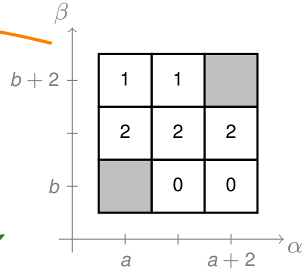
$$Bx_B \le b_B - \beta g$$
$$f^\top x_B = \alpha$$
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

Linear *pattern:* One feasible residue per (α, β)-pair
$\implies$ residue is linear in α and β.

Reduction to

$$Ax_A + \alpha e \le b_A$$
$$h^\top x_A - \beta = 0$$
$$(\alpha, \beta) \in \Pi$$
$$\gamma_A^\top x_A + r_1\alpha + r_2\beta + r_0 \equiv 2 \pmod{3}$$

Single lower-dimensional problem ✓



Studying patterns IV

(for $m = 3$ and $r = 2$)

CCTU feasibility problem

$$\begin{pmatrix} A & ef^\top \\ gh^\top & B \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} b_A \\ b_B \end{pmatrix}$$

$$\gamma_A^\top x_A + \gamma_B^\top x_B \equiv 2 \pmod{3}$$

A-subproblem

$$A x_A \leq b_A - \alpha e$$

$$h^\top x_A = \beta$$

$$\gamma_A^\top x_A \equiv r_A \pmod{3}$$

B-subproblem

$$B x_B \leq b_B - \beta g$$

$$f^\top x_B = \alpha$$

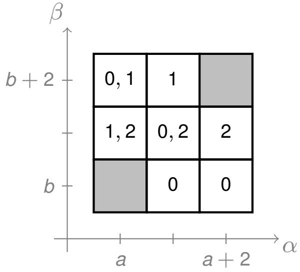
$$\gamma_B^\top x_B \equiv r_B \pmod{3}$$

For parameters $\alpha := f^\top x_B$ and $\beta := h^\top x_A$, we want solutions with $r_A + r_B \equiv 2 \pmod{3}$.

***Mixed** pattern:*

- Combine previous ideas + extra insights
- Reduce to
  - $\rightarrow$ at most one smaller-dimensional problem
  - $\rightarrow$ constantly many easier problems

✓



Solving base block problems

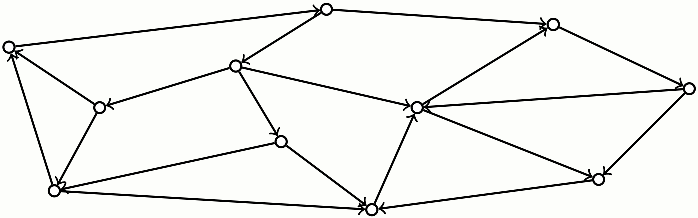
Network matrices and their transposes

Network matrices

Theorem: Network matrix problems

$\exists$ strongly poly. randomized alg. for CCTU problems with unary encoded objectives, constant m and network constraint matrices.

- Reduction to congruency-constrained circulation problems
- Examples:
  - $m = 2 \rightarrow$ Find a shortest odd cycle.
  - $m = 3 \rightarrow$ Find a shortest circulation using $1 \pmod 3$ many edges.

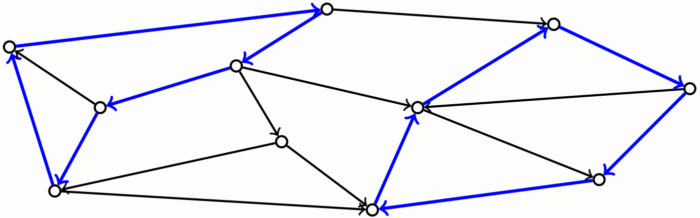


Network matrices

Theorem: Network matrix problems

$\exists$ strongly poly. randomized alg. for CCTU problems with unary encoded objectives, constant m and network constraint matrices.

- Reduction to congruency-constrained circulation problems
- Examples:
  - $m = 2 \rightarrow$ Find a shortest odd cycle.
  - $m = 3 \rightarrow$ Find a shortest circulation using $1 \pmod 3$ many edges.

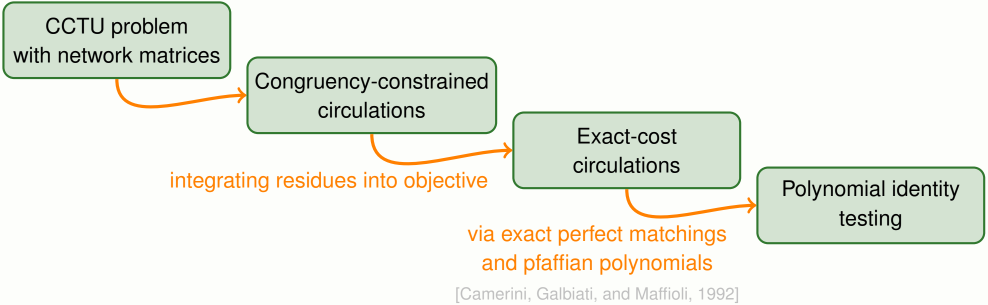


Network matrices

Theorem: Network matrix problems

∃ strongly poly. randomized alg. for CCTU problems with unary encoded objectives, constant m and network constraint matrices.

- Our approach:



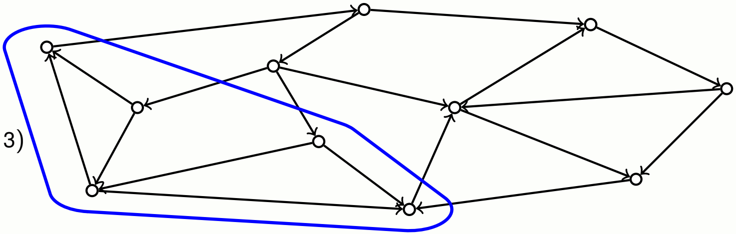
Transposes of network matrices

Theorem: Transposed network matrix problems

∃ strongly poly. alg. for CCTU problems with constant prime power modulus m and transposed network constraint matrices.

- Reduction to congruency-constrained directed minimum cut problems

$|C| \equiv 2 \pmod 3$

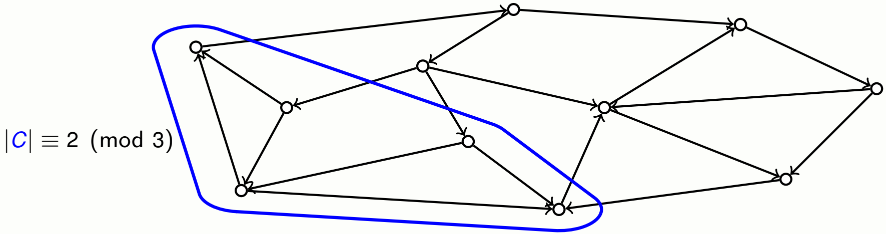


Transposes of network matrices

Theorem: Transposed network matrix problems

∃ strongly poly. alg. for CCTU problems with constant prime power modulus m and transposed network constraint matrices.

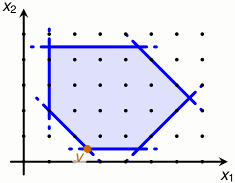
- Reduction to congruency-constrained directed minimum cut problems



- Efficient algorithms known for prime power moduli [N., Sudakov, and Zenklusen, 2018]
- Undirected: Randomized approximation scheme for arbitrary modulus [N. and Zenklusen, 2019]

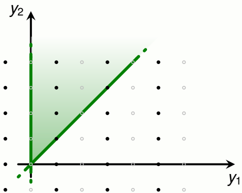
Open questions

Open questions



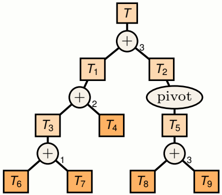
Δ-modular integer programming

$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.



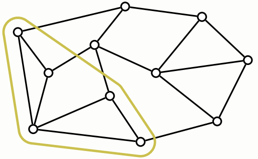
CCTU

$\min\{\tilde{c}^\top y \colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.



Seymour's TU decomposition

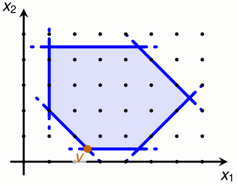
Reduction to congruency-constrained base block problems.



Base block problems

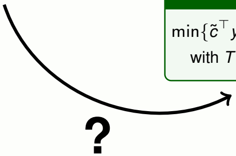
Interpretation as congruency-constrained cut and circulation problems

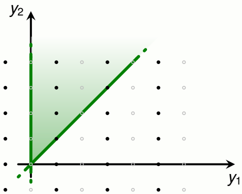
Open questions



Δ-modular integer programming

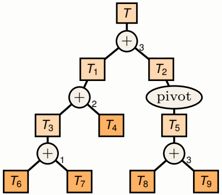
$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.





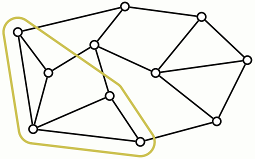
CCTU

$\min\{\tilde{c}^\top y \colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.



Seymour's TU decomposition

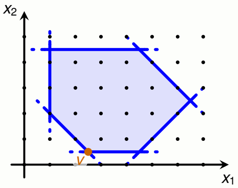
Reduction to congruency-constrained base block problems.



Base block problems

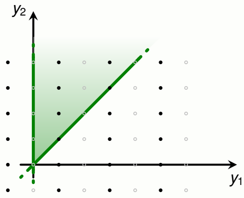
Interpretation as congruency-constrained cut and circulation problems

Open questions



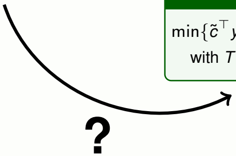
Δ-modular integer programming

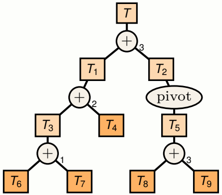
$\min\{c^\top x : Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.



CCTU

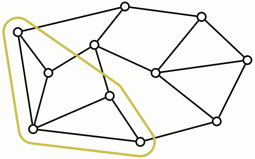
$\min\{\tilde{c}^\top y : Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.





Seymour's TU decomposition

Reduction to congruency-constrained base block problems.

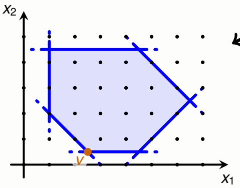


Base block problems

Interpretation as congruency-constrained cut and circulation problems

- Transformation for conic problems?
- How to deal with non-tight constraints?

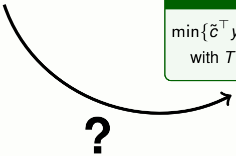
Open questions

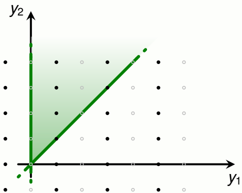


Structural results?

Δ-modular integer programming

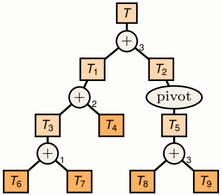
$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.





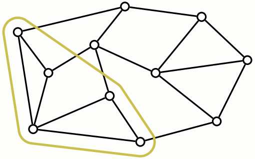
CCTU

$\min\{\tilde{c}^\top y \colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.



Seymour's TU decomposition

Reduction to congruency-constrained base block problems.

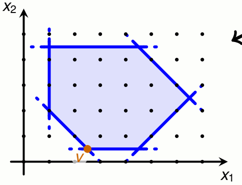


Base block problems

Interpretation as congruency-constrained cut and circulation problems

- Transformation for conic problems?
- How to deal with non-tight constraints?

Open questions

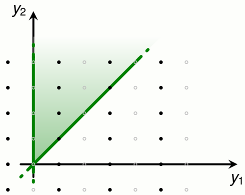


Structural results?

Δ-modular integer programming

$\min\{c^\top x\colon Ax \leqslant b, x \in \mathbb{Z}^n\}$

wh. A is Δ-modular, fract. relaxation.



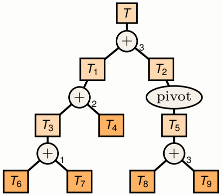
CCTU

$\min\{\tilde{c}^\top y\colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$

with T totally unimodular, $m = \Delta$.

?

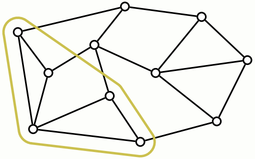
- Transformation for conic problems?
- How to deal with non-tight constraints?



Seymour's TU decomposition

Reduction to congruency-constrained base block problems.

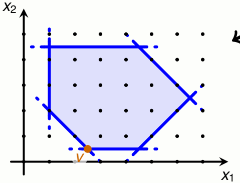
- Optimization?
- Beyond $m = 3$?



Base block problems

Interpretation as congruency-constrained cut and circulation problems

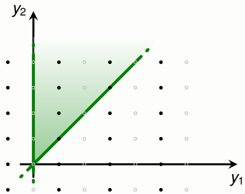
Open questions



Structural results?

Δ-modular integer programming

$\min\{c^\top x \colon Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.

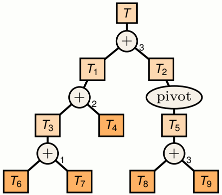


CCTU

$\min\{\tilde{c}^\top y \colon Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.

?

- Transformation for conic problems?
- How to deal with non-tight constraints?

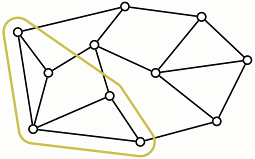


Seymour's TU decomposition

Reduction to congruency-constrained base block problems.

- Optimization?
- Beyond $m = 3$?

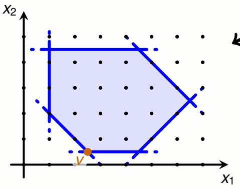
- Beyond $m = p^\alpha$ for cuts?
- Deterministic approach for circulations?



Base block problems

Interpretation as congruency-constrained cut and circulation problems

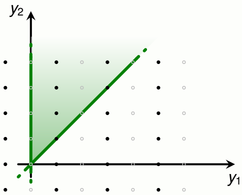
Open questions



Structural results?

Δ-modular integer programming

$\min\{c^\top x : Ax \leqslant b, x \in \mathbb{Z}^n\}$
wh. A is Δ-modular, fract. relaxation.



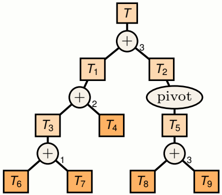
CCTU

$\min\{\tilde{c}^\top y : Ty \leqslant b, \gamma^\top y \equiv r \pmod{m}\}$
with T totally unimodular, $m = \Delta$.

?

- Transformation for conic problems?
- How to deal with non-tight constraints?

Do we need to go through Seymour's decomposition?

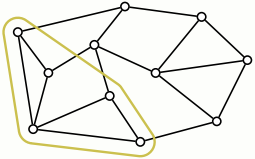


Seymour's TU decomposition

Reduction to congruency-constrained base block problems.

- Optimization?
- Beyond $m = 3$?

- Beyond $m = p^\alpha$ for cuts?
- Deterministic approach for circulations?



Base block problems

Interpretation as congruency-constrained cut and circulation problems