A Better-Than-1.6-Approximation for Prize-Collecting TSP

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NSF
Swiss National
Science Foundation
DFG Deutsche
Forschungsgemeinschaft
German Research Foundation

Prize-Collecting TSP

Input:

Complete $G = (V, E)$, root $r \in V$.

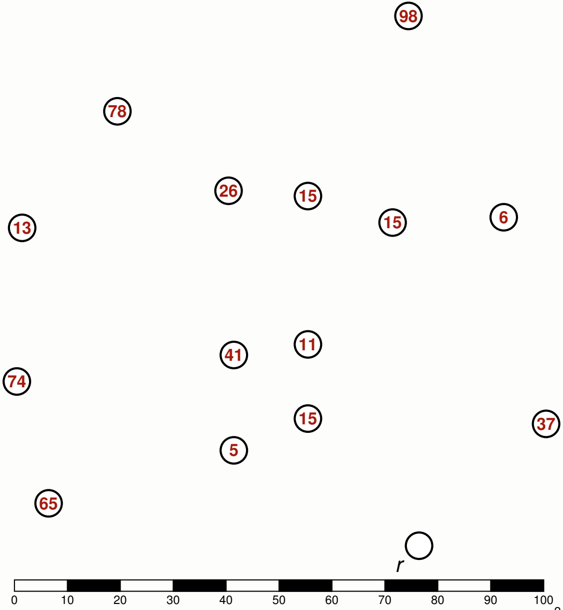
Metric $c\colon E \to \mathbb{R}_{\geq 0}$, penalties $\pi\colon V \setminus \{r\} \to \mathbb{R}_{\geq 0}$.

Task:

Find cycle $C = (V_C, E_C)$ in G covering r minimizing

$$\sum_{e \in E_C} c_e + \sum_{v \in V \setminus V_C} \pi_v \ .$$

connection cost *penalty cost*



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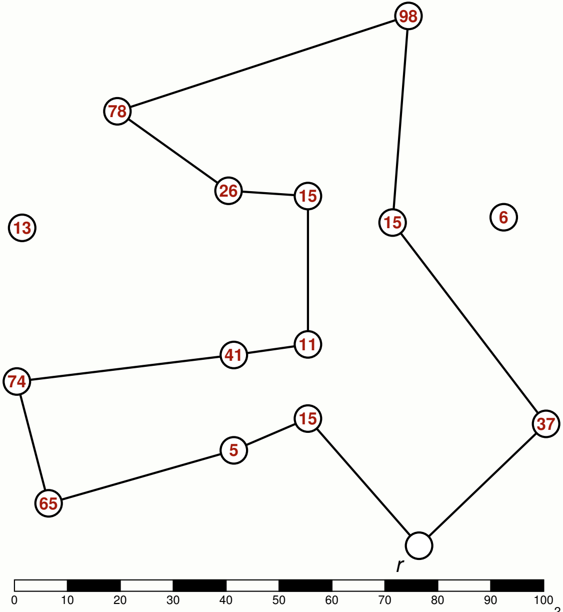
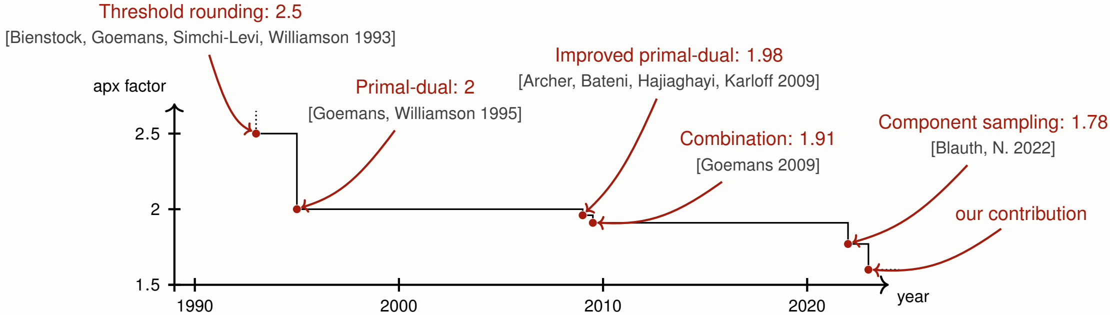


Approximation algorithms



Main result

There is a polynomial time 1.599-approximation algorithm for PCTSP.

[Blauth, Klein, N. 2023+]

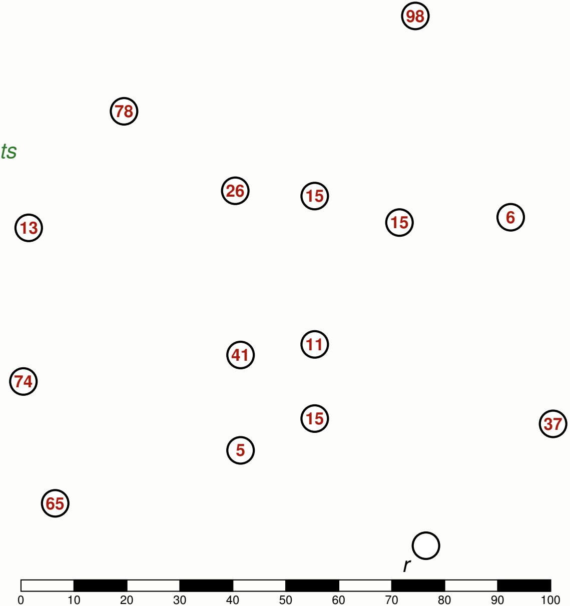
An LP relaxation

degree constraints

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e + \sum_{v \in V} \pi_v (1 - y_v) \\
x(\delta(v)) &= 2y_v && \forall v \in V \setminus \{r\} \\
x(\delta(r)) &\leqslant 2 \\
x(\delta(S)) &\geqslant 2y_v && \forall S \subseteq V \setminus \{r\}, v \in S \\
y_r &= 1 \\
x_e &\geqslant 0 && \forall e \in E \\
y_v &\geqslant 0 && \forall v \in V
\end{aligned}
$$

cut constraints

vertex connectivity



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$$\begin{array}{lrcll}
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degree constraints

cut constraints

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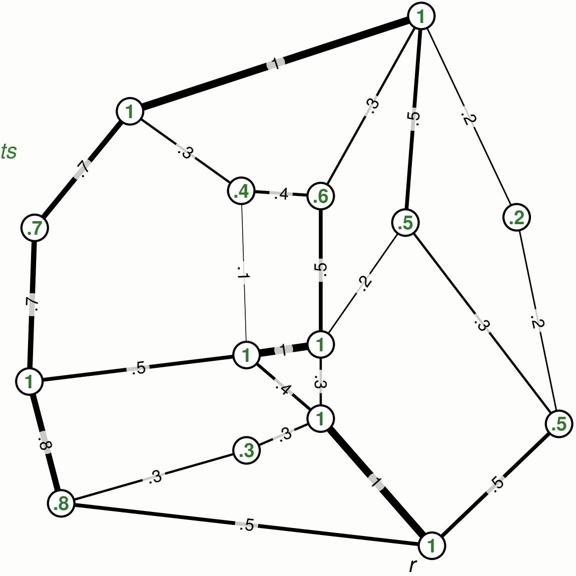


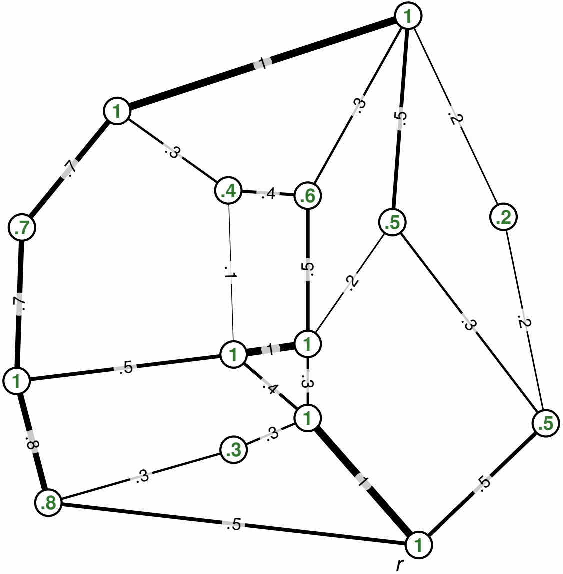
Sampling trees from LP solutions

Lemma (similar to [Bang-Jensen, Frank, Jackson 1995])

Given an LP solution (x^*, y^*), we can sample a tree T with

(i) $\mathbb{E}[c(E[T])] \leq c^\top x^*$, and

(ii) $\mathbb{P}[v \in V[T]] = y_v^*$ for each $v \in V$.



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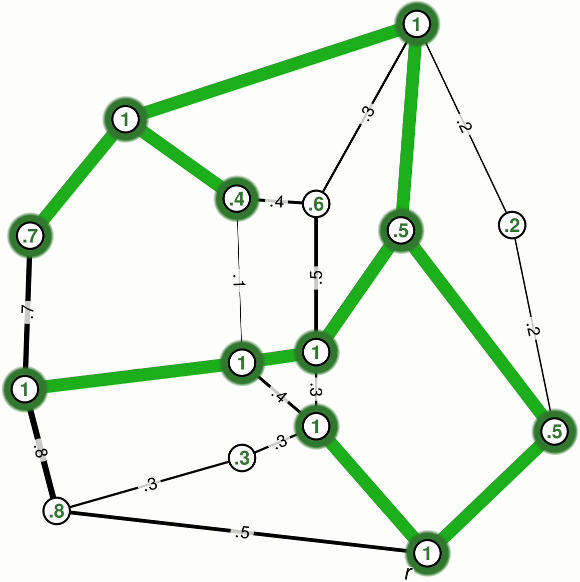

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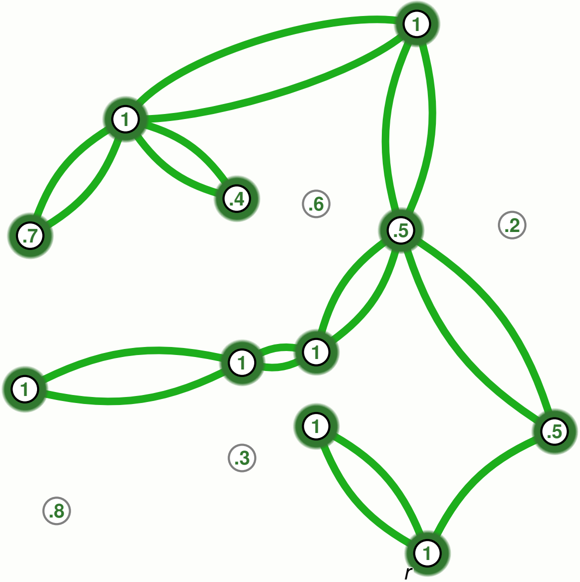
Simple 2-approximation

Sample a tree, double its edges, get expected cost at most

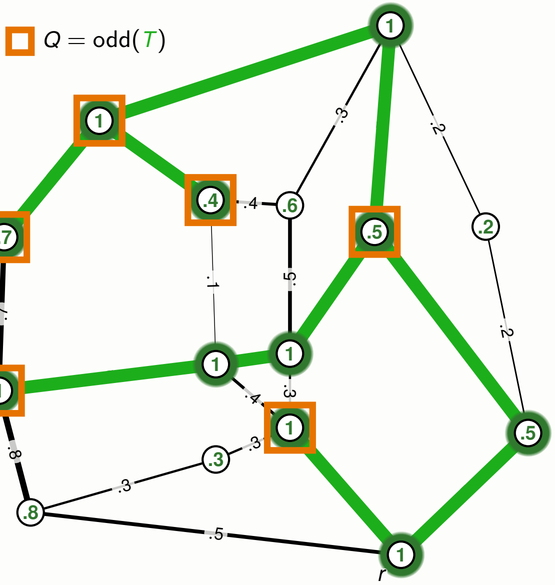
$$2c^\top x^* + \pi^\top (1 - y^*) \ .$$



Bounding the cost of parity correction

Dominant of the Q-join polytope

$$P^{\uparrow}_{Q\text{-join}} = \left\{ x \in \mathbb{R}^E_{\geq 0} : x(\delta(S)) \geq 1 \quad \begin{array}{c}\text{for } S \subseteq V \text{ s.t.}\\ |S \cap Q| \text{ odd}\end{array} \right\}$$



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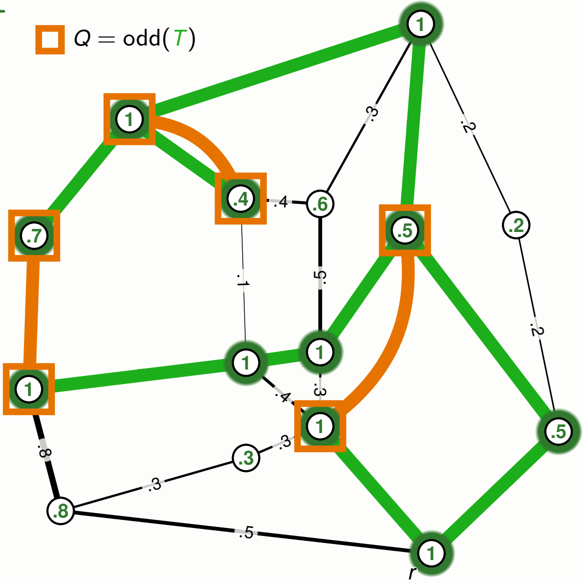
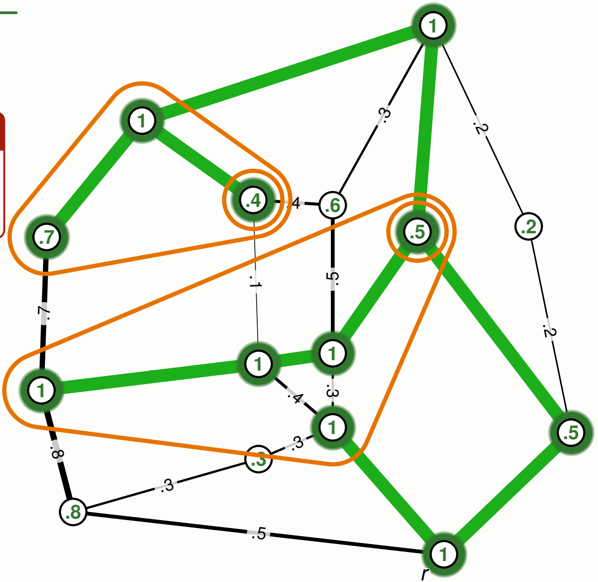


Bounding the cost of parity correction

Dominant of the Q-join polytope for $Q = \mathrm{odd}(T)$

$$P^{\uparrow}_{Q\text{-join}} = \left\{ x \in \mathbb{R}^E_{\geq 0} : x(\delta(S)) \geq 1 \quad \begin{matrix} \text{for } S \subseteq V \text{ s.t.} \\ |\delta(S) \cap T| \text{ odd} \end{matrix} \right\}$$



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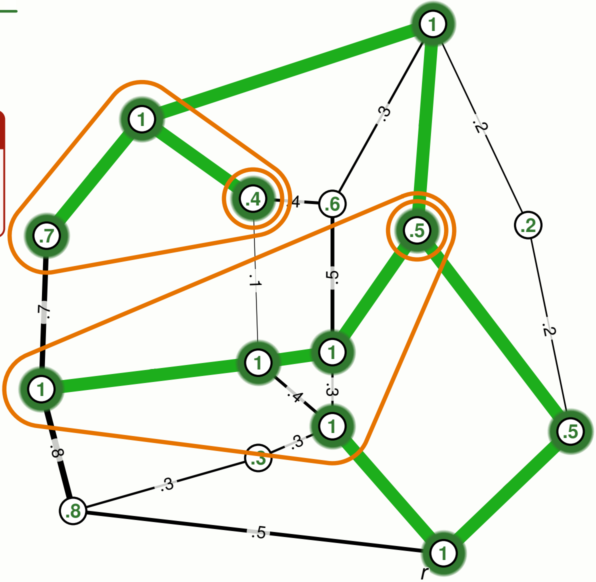
- Standard TSP: $\frac{1}{2}x^* \in P^{\uparrow}_{\mathrm{odd}(T)\text{-join}}$ for *any* T



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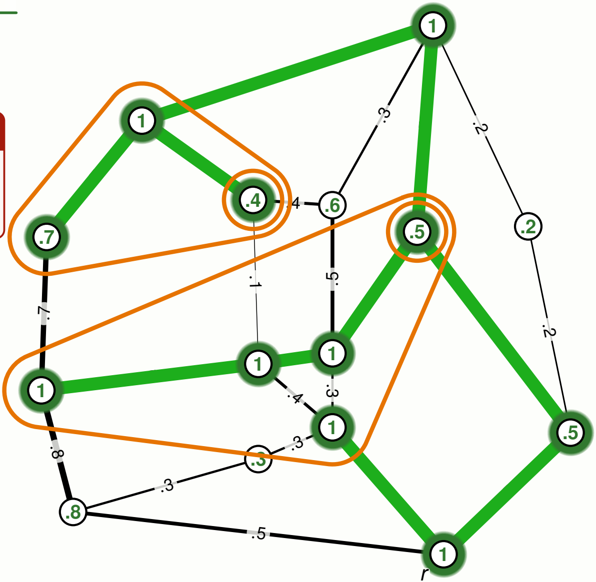
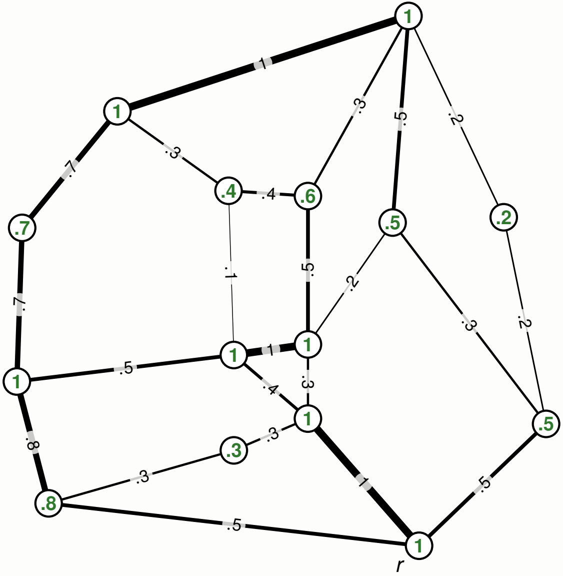
- Standard TSP: $\frac{1}{2}x^* \in P^{\uparrow}_{\mathrm{odd}(T)\text{-join}}$ for *any* T
- PCTSP: naïve scaling requires extra factor

$$\frac{1}{\min_{v \in T} y_v}$$



Our algorithm

1. **Preprocessing:** Drop all vertices $v \in V$ with $y_v < \delta$ via **splitting off**.



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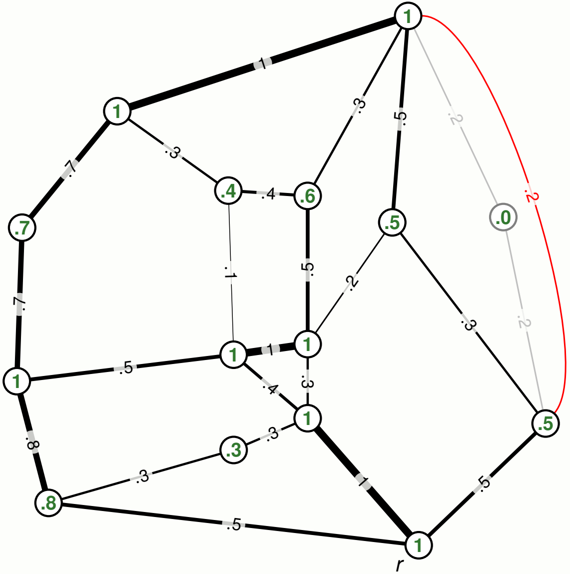
Example: $\delta = .3$



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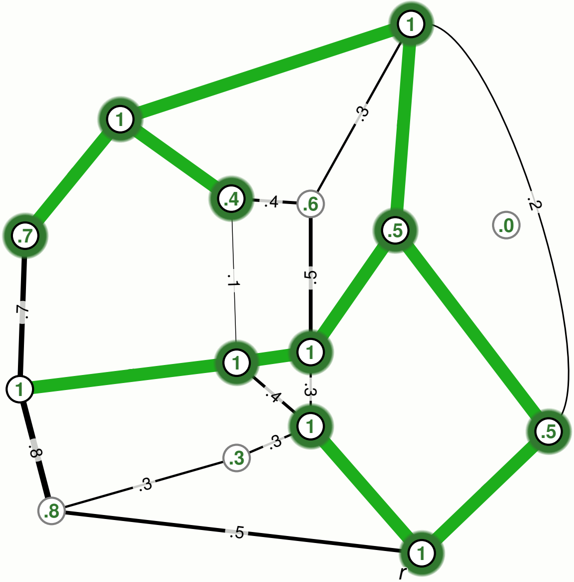

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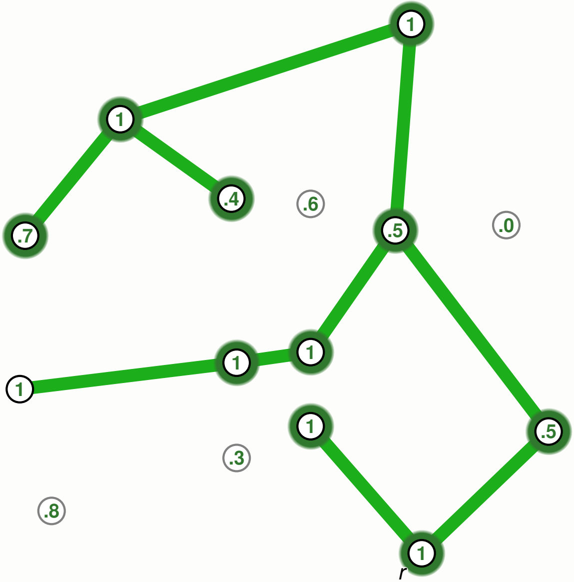

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1. **Preprocessing:** Drop all vertices $v \in V$ with $y_v < \delta$ via **splitting off**.
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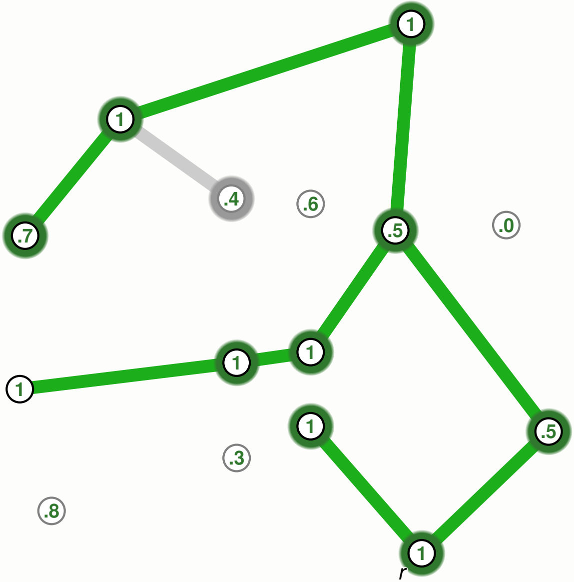
3. **Pruning:** Cut subtrees that only contain vertices v with $y_v < \gamma$.

Example: $\gamma = .6$



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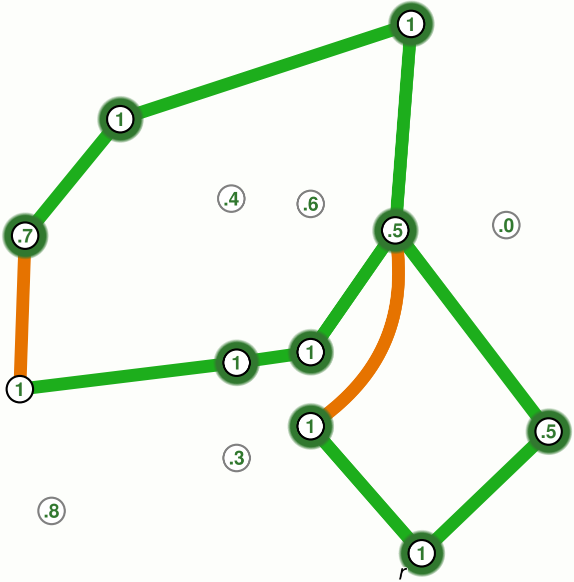

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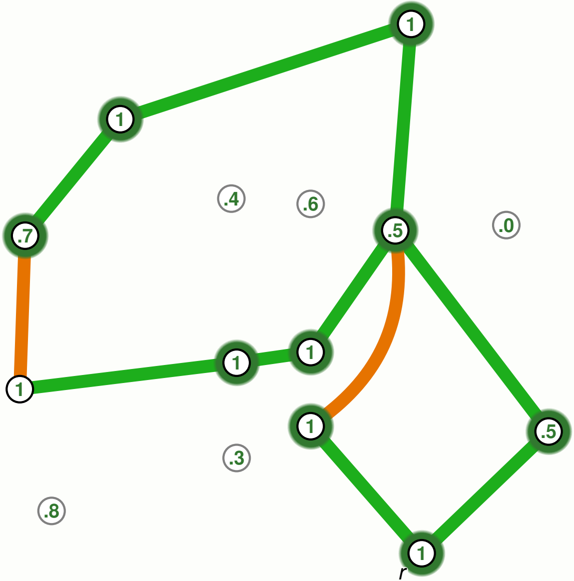
Preprocessing and **Pruning** yield improved bound on cost of parity correction while paying more penalties.



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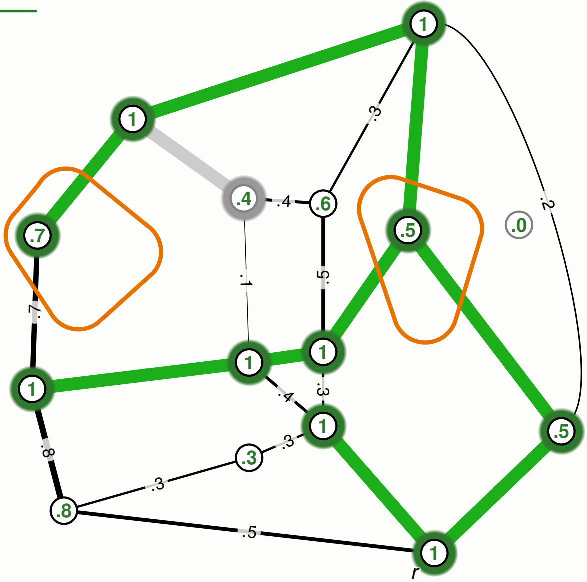

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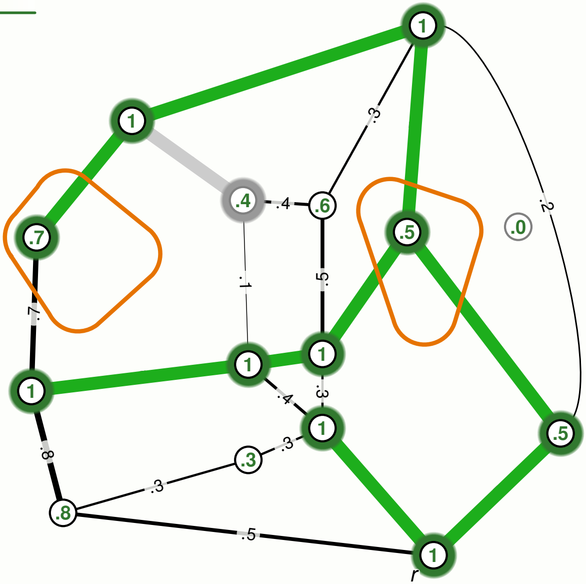

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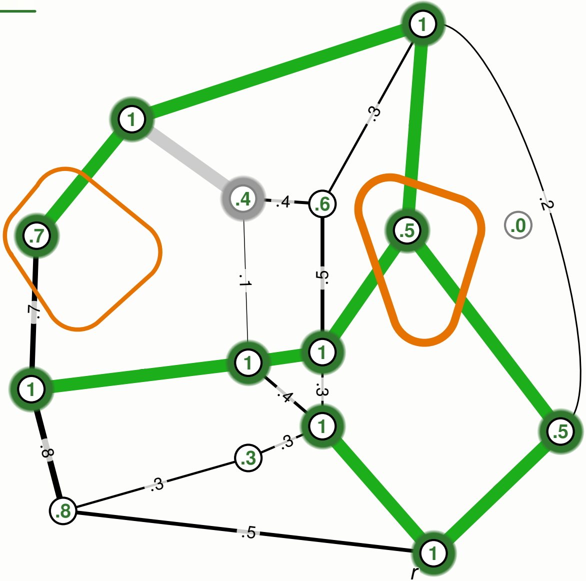

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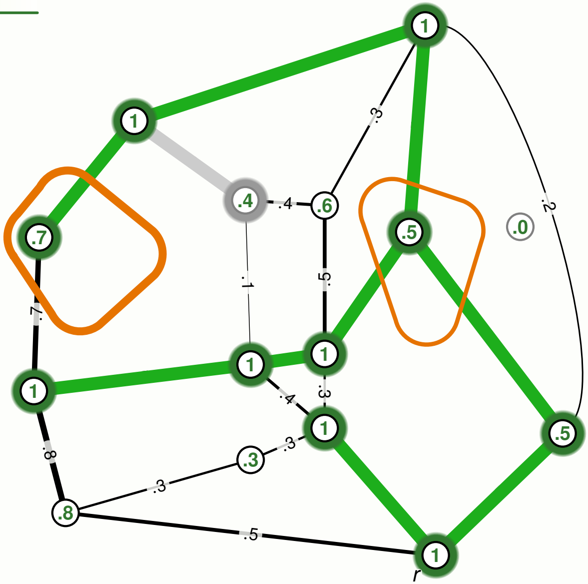

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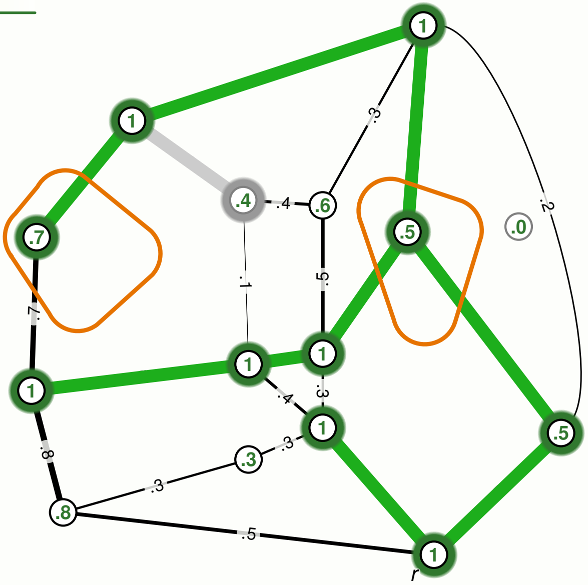
Improves over
$2\alpha\delta + \beta$
w/o pruning!



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$$z(\delta(S)) \geq \min\{\, 2\alpha\delta + 3\beta \,,\, 2\alpha\gamma + \beta \,\} \overset{!}{\geq} 1$$

$$\uparrow$$

Improves over

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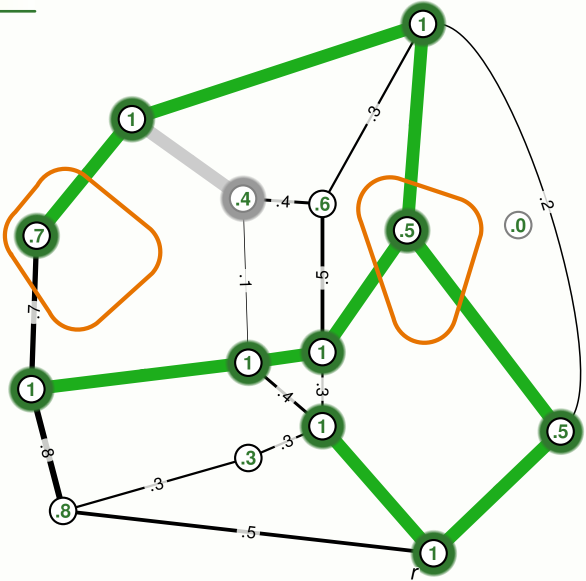
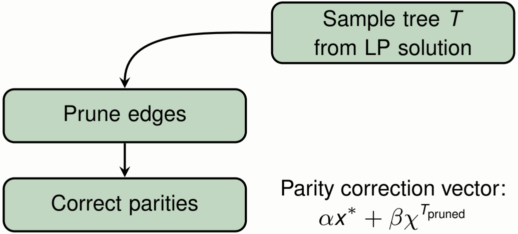


A constant thresholds algorithm



Sample tree T from LP solution

Prune edges

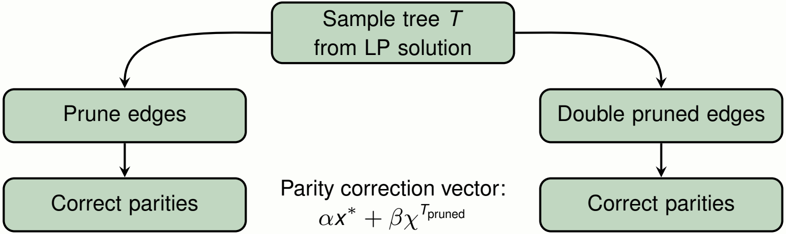
Correct parities

Parity correction vector:
$\alpha x^* + \beta \chi^{T_{\text{pruned}}}$

Exp. conn. cost

$$(1+\alpha+\beta)\, c^\top x^* - (1+\beta)\,\mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

A constant thresholds algorithm



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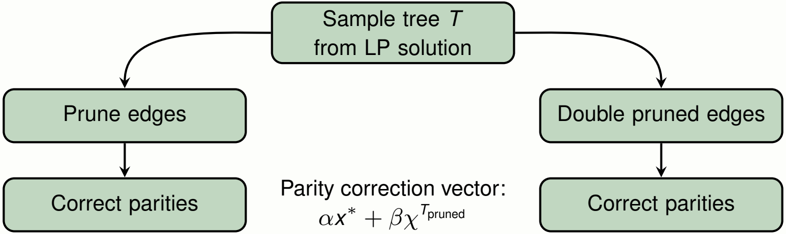

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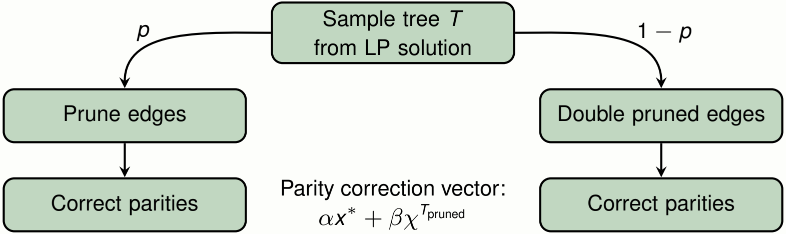

Parity correction vector: $\alpha x^* + \beta \chi^{T_{\text{pruned}}}$

Exp. conn. cost	$(1+\alpha+\beta)\, c^\top x^* - (1+\beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$	$(1+\alpha+\beta)\, c^\top x^* + (1-\beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$

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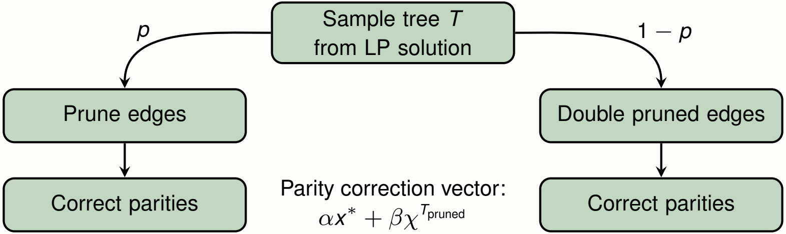

Sample tree T from LP solution

p

$1-p$

Prune edges

Double pruned edges

Correct parities

Correct parities

Parity correction vector:
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Exp. conn. cost

$$(1+\alpha+\beta)\,c^\top x^* - (1+\beta)\,\mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

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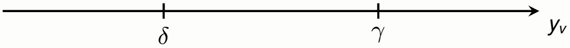
Exp. penalty cost



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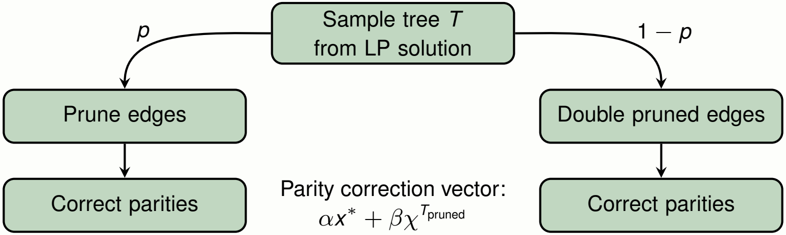

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$1-p$

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Double pruned edges

Correct parities

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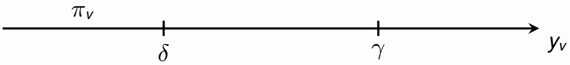

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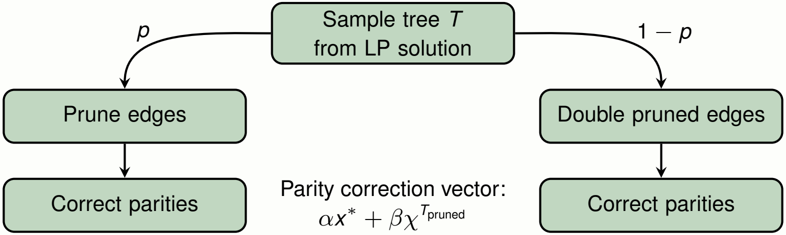

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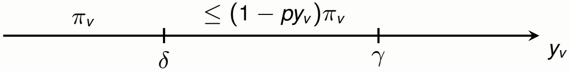
Exp. conn. cost

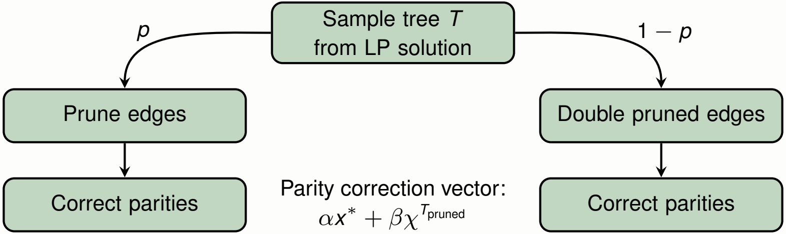
$$(1 + \alpha + \beta)\, c^\top x^* - (1 + \beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

$$(1 + \alpha + \beta)\, c^\top x^* + (1 - \beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

Exp. penalty cost



A constant thresholds algorithm



Sample tree T from LP solution

p

$1 - p$

Prune edges

Double pruned edges

Correct parities

Correct parities

Parity correction vector:
$\alpha x^* + \beta \chi^{T_{\text{pruned}}}$

Exp. conn. cost

$$(1 + \alpha + \beta)\, c^\top x^* - (1 + \beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

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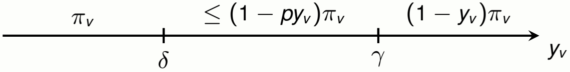
Exp. penalty cost



π_v

$\leq (1 - p y_v)\pi_v$

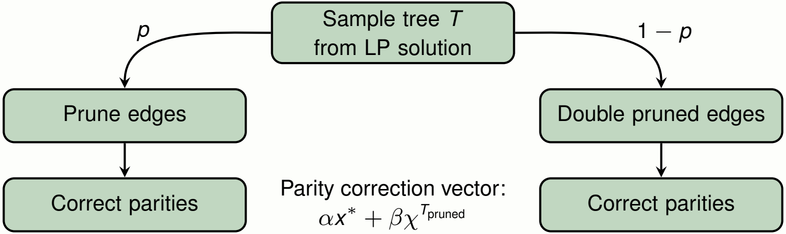
$(1 - y_v)\pi_v$

δ

γ

y_v

A constant thresholds algorithm



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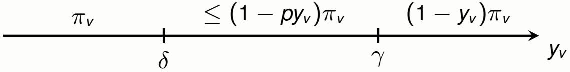
$$(1+\alpha+\beta)\, c^\top x^* - (1+\beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

$$(1+\alpha+\beta)\, c^\top x^* + (1-\beta)\, \mathbb{E}[c(T \setminus T_{\text{pruned}})]$$

Exp. penalty cost



Result

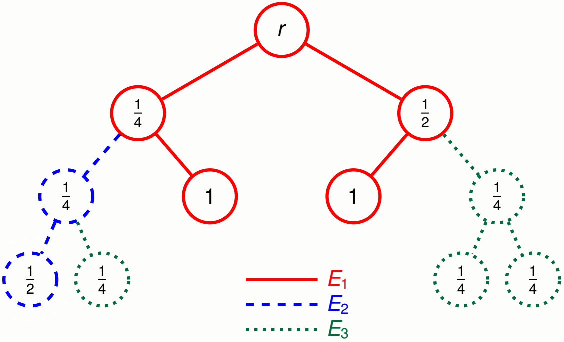
Approx. ratio $\max\left\{1+\alpha+\beta, \frac{1}{1-\delta}, \frac{1-p\gamma}{1-\gamma}\right\} \approx 1.72$ for optimized parameter choice.

Randomized improvements

Idea: Random pruning threshold γ

edge e is pruned with high probability

$\implies$ large budget for parity correction if e is not pruned



Randomized improvements

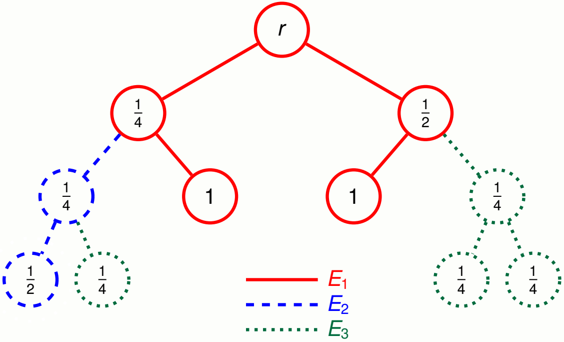
Idea: Random pruning threshold γ

edge e is pruned with high probability

$\implies$ large budget for parity correction if e is not pruned

Parity correction vector:

$$z = \alpha \cdot x^* + \sum_{i:\ \eta_i \geq \gamma} \beta_i \chi^{E_i}$$



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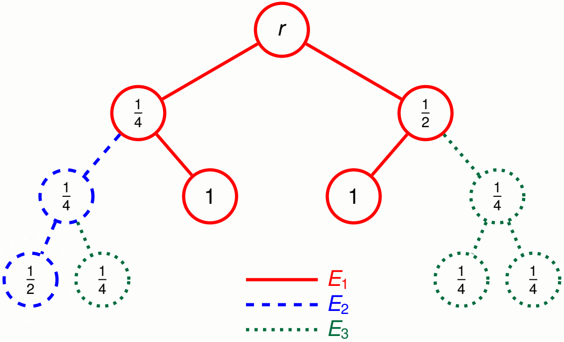

Result

ρ-approximation via simple analysis for

$$\rho = \frac{1+\sqrt{5}}{2} \approx 1.618\ .$$

Randomized improvements

Idea: Random pruning threshold γ

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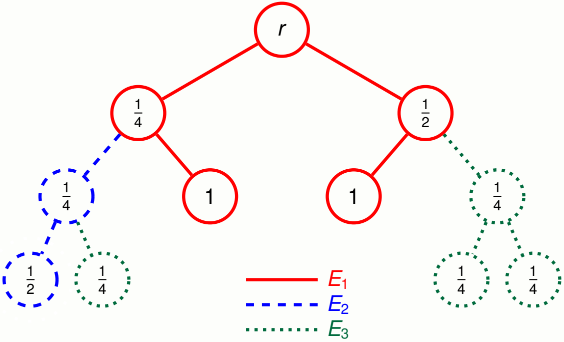


Improvement to $\rho = 1.599$:
randomize choice of δ

Result

ρ-approximation via simple analysis for

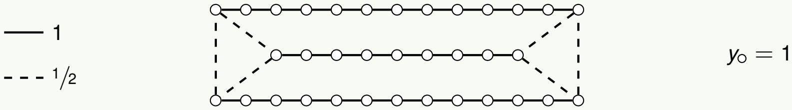
$$\rho = \frac{1+\sqrt{5}}{2} \approx 1.618\ .$$

Derandomization:
try $\delta, \gamma \in \{y_v : v \in V\}$

Open questions

Lower bound on the integrality gap

The integrality gap of the natural LP relaxation for prize-collecting TSP is at least $^4/_3$.



- Can we improve the best known **lower bound** on the integrality gap of the natural LP relaxation?
- Can we close the gap between the **approximability** of prize-collecting TSP and metric TSP?