An improved approximation guarantee for
Prize-Collecting TSP

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Swiss National Science Foundation

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Prize-Collecting TSP

Input:

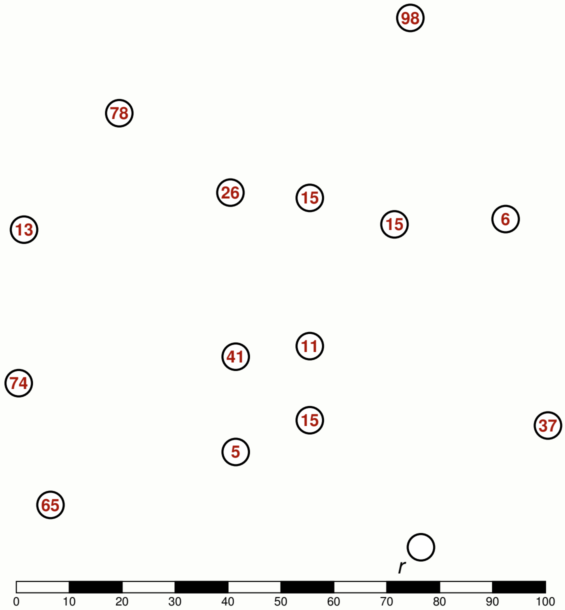
Complete $G = (V, E)$, root $r \in V$.

Metric $c\colon E \to \mathbb{R}_{\geq 0}$, penalties $\pi\colon V \setminus \{r\} \to \mathbb{R}$.

Task:

Find cycle $C = (V_C, E_C)$ in G covering r minimizing

$$\sum_{e \in E_C} c_e + \sum_{v \in V \setminus V_C} \pi_v \ .$$



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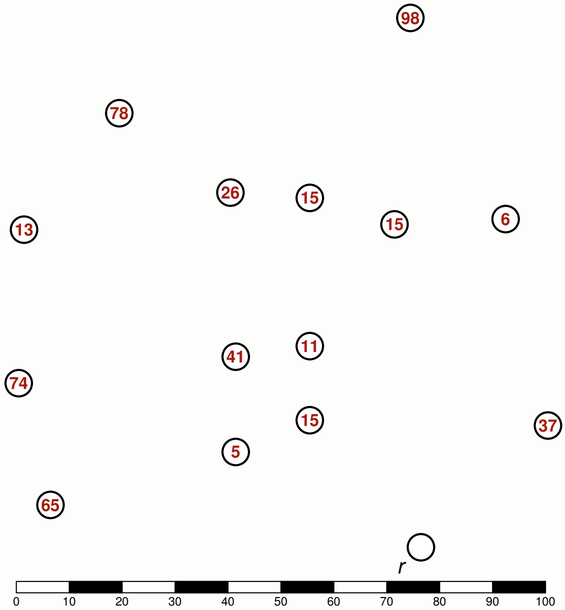
$$\underbrace{\sum_{e \in E_C} c_e}_{\text{connection cost}} + \underbrace{\sum_{v \in V \setminus V_C} \pi_v}_{\text{penalty cost}} \ .$$

connection cost *penalty cost*



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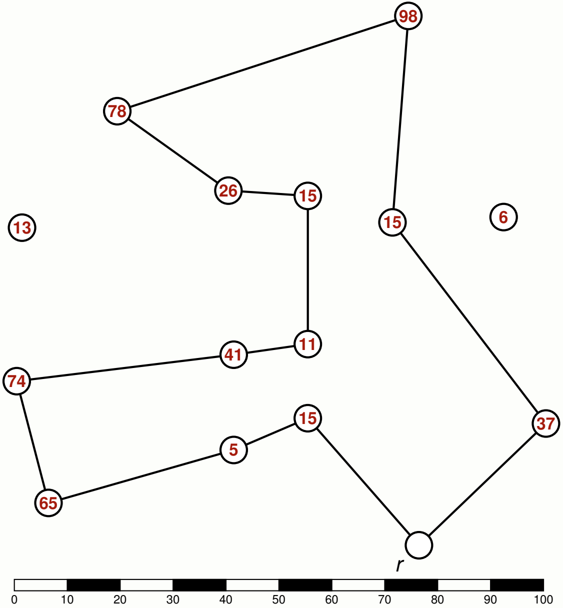
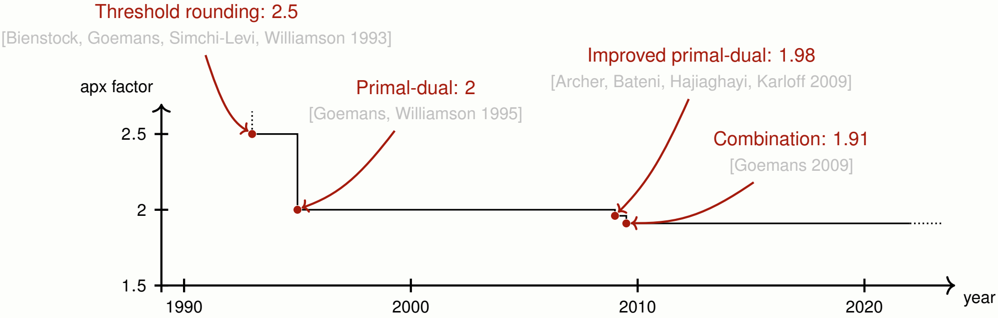


Approximation algorithms



Threshold rounding: 2.5

[Bienstock, Goemans, Simchi-Levi, Williamson 1993]

Primal-dual: 2

[Goemans, Williamson 1995]

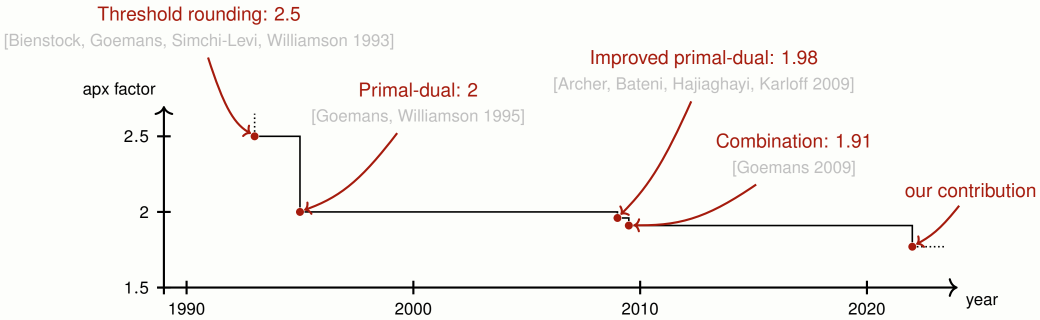
Improved primal-dual: 1.98

[Archer, Bateni, Hajiaghayi, Karloff 2009]

Combination: 1.91

[Goemans 2009]

Approximation algorithms



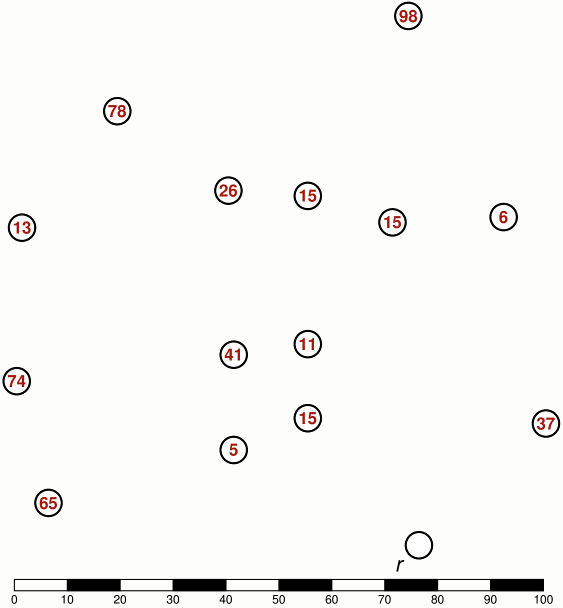
Theorem

There is an efficient 1.774-approximation algorithm for PCTSP.

[Blauth, Nägele 2022]

An LP formulation

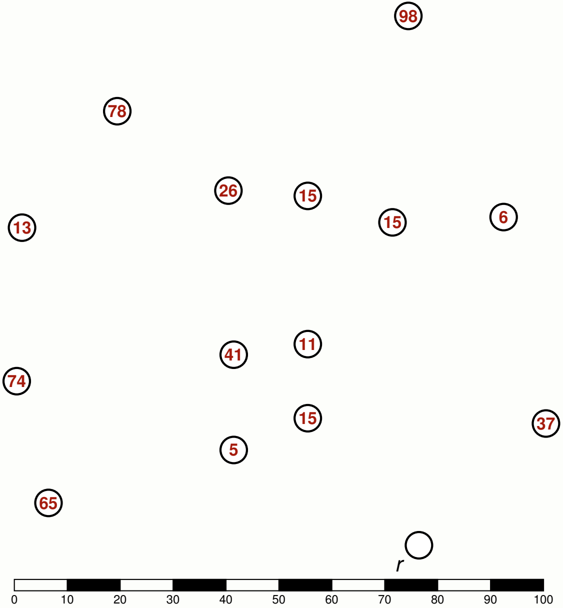
$$\begin{array}{rcll} \min & \displaystyle\sum_{e\in E} c_e x_e + \sum_{v\in V} \pi_v (1 - y_v) & & \\ x(\delta(v)) & = & 2y_v & \forall v \in V \setminus \{r\} \\ x(\delta(r)) & \leqslant & 2 & \\ x(\delta(S)) & \geqslant & 2y_v & \forall S \subseteq V \setminus \{r\}, v \in S \\ y_r & = & 1 & \\ x_e & \geqslant & 0 & \forall e \in E \\ y_v & \in & [0,1] & \forall v \in V \end{array}$$



An LP formulation

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e + \sum_{v \in V} \pi_v (1 - y_v) \\ x(\delta(v)) &= 2y_v \qquad \forall v \in V \setminus \{r\} \\ x(\delta(r)) &\leqslant 2 \\ x(\delta(S)) &\geqslant 2y_v \qquad \forall S \subseteq V \setminus \{r\}, v \in S \\ y_r &= 1 \\ x_e &\geqslant 0 \qquad \forall e \in E \\ \underset{\uparrow}{y_v} &\in [0,1] \qquad \forall v \in V \end{aligned}$$

vertex connectivity



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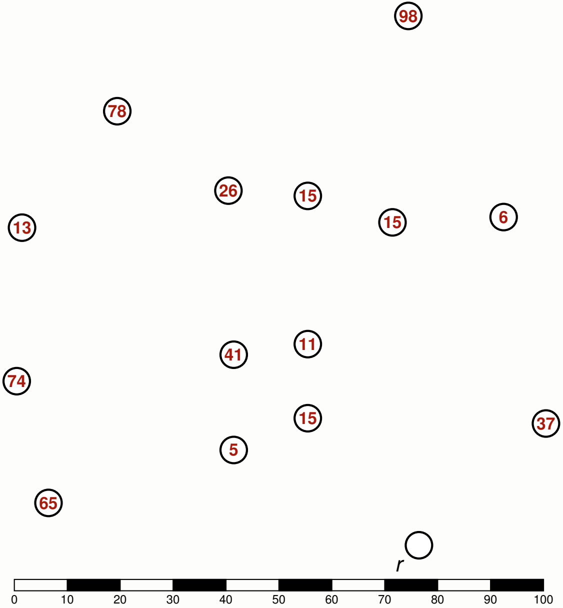
cut constraints

vertex connectivity



An LP formulation

degree constraints

$$
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\min & \multicolumn{4}{l}{\displaystyle\sum_{e \in E} c_e x_e + \sum_{v \in V} \pi_v (1 - y_v)} \\
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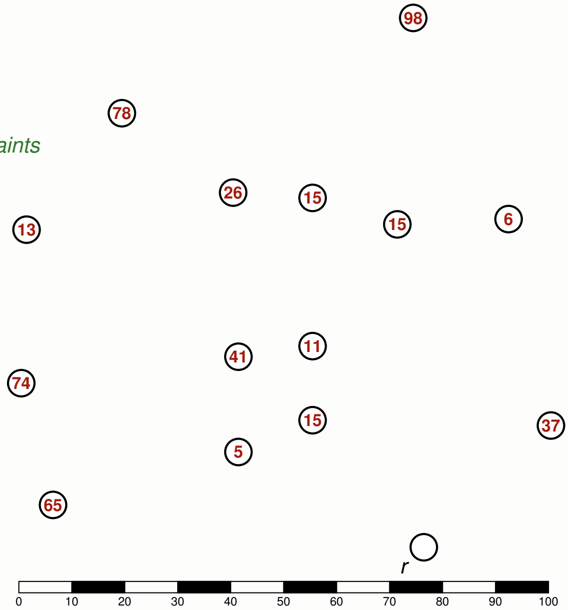

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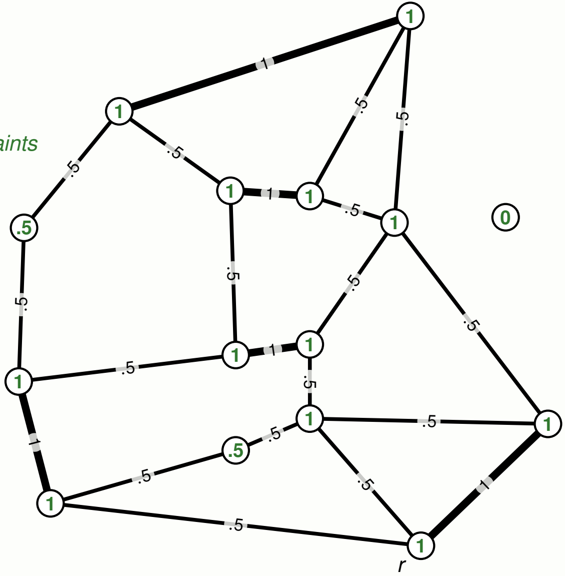

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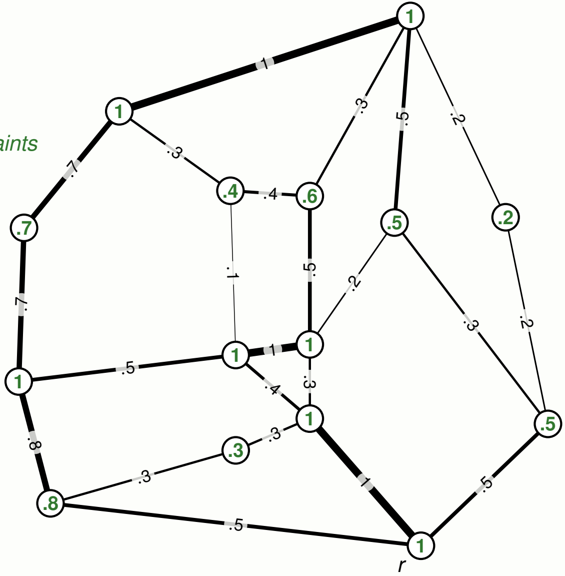

Threshold rounding

Threshold rounding with threshold γ

Apply Christofides on $V_\gamma := \{v \in V \colon y_v \geqslant \gamma\}$.

$$
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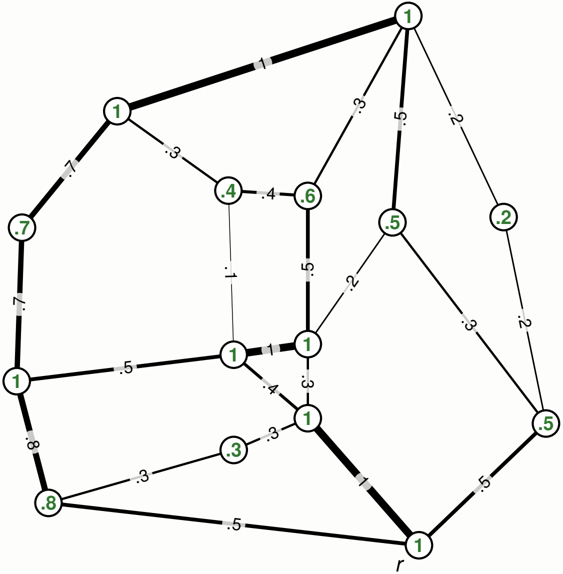

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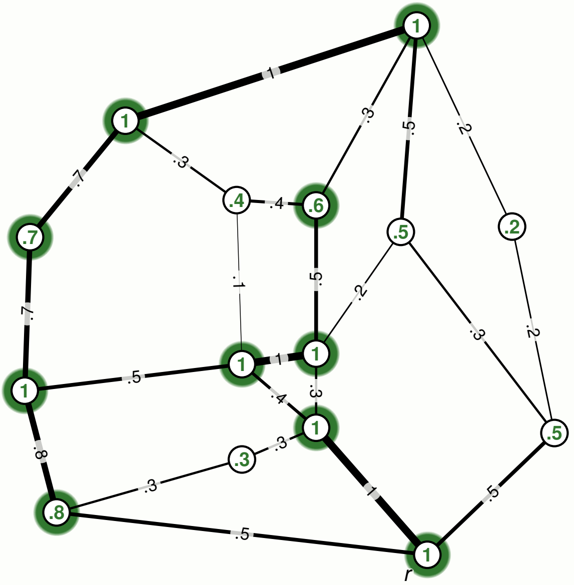
Example: $\gamma = {}^{3}\!/_{5}$



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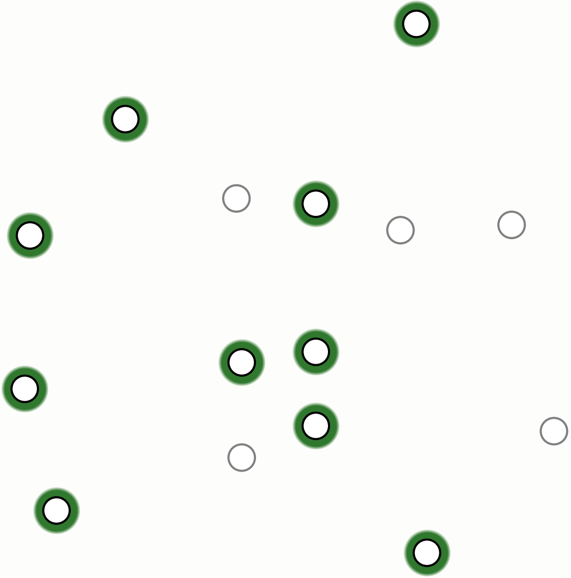

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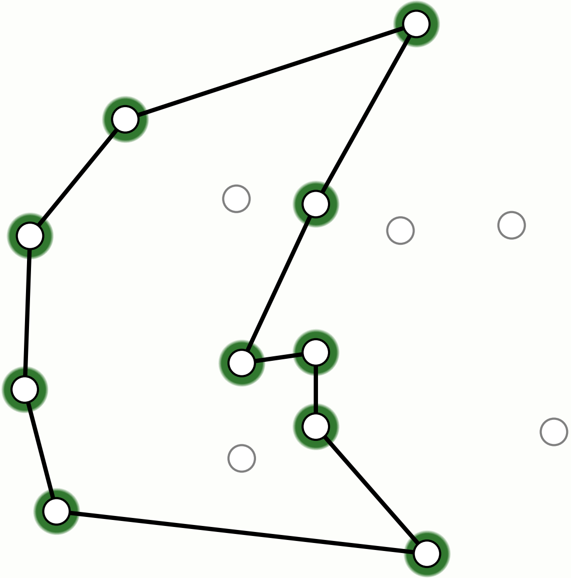
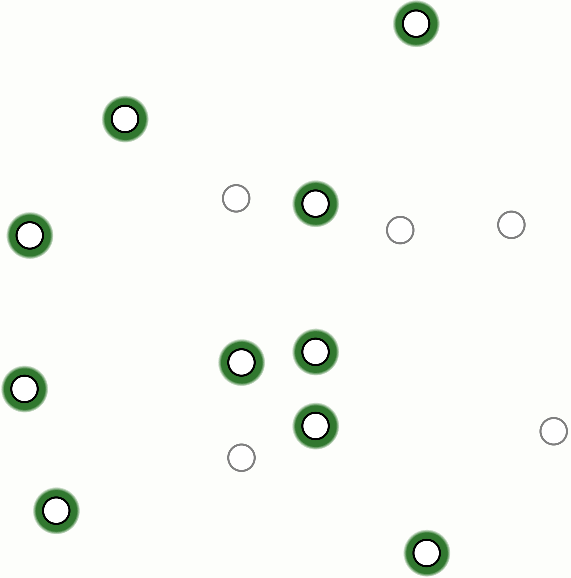


Recalling Christofides' Algorithm

Christofides' Algorithm

1. Take a shortest spanning tree T.
2. Add a shortest $\mathrm{odd}(T)$-join J.
3. Take an Eulerian tour, shortcut.



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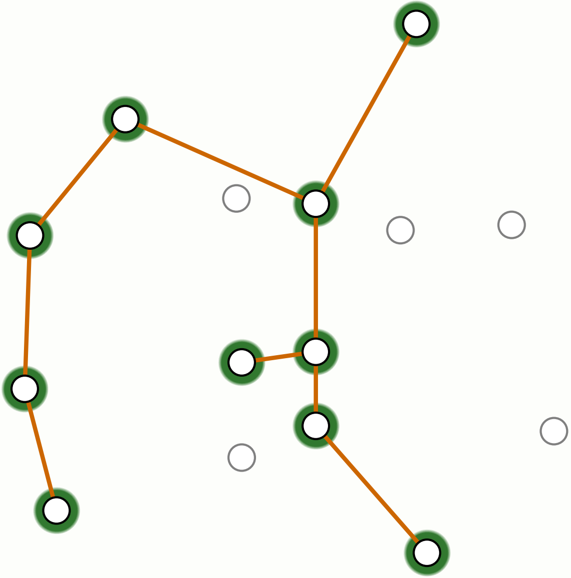

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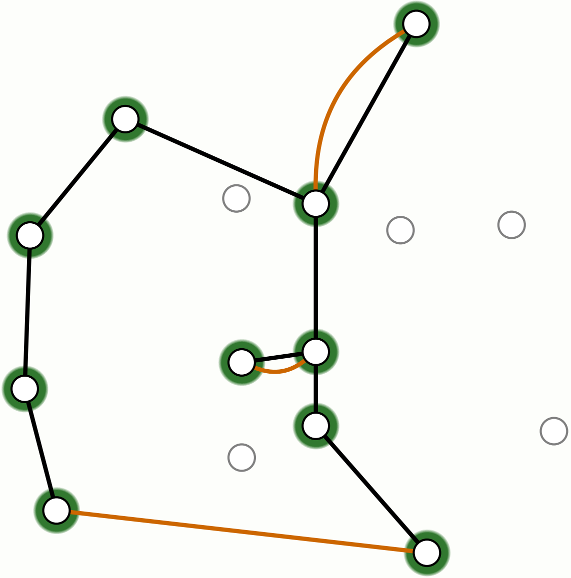

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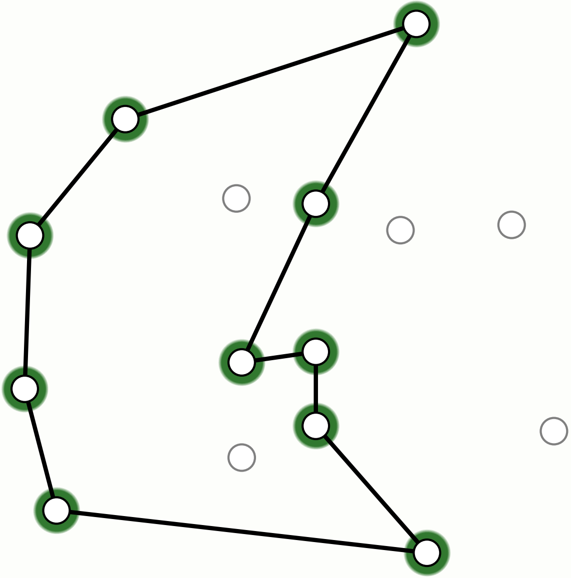

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Held-Karp relaxation:

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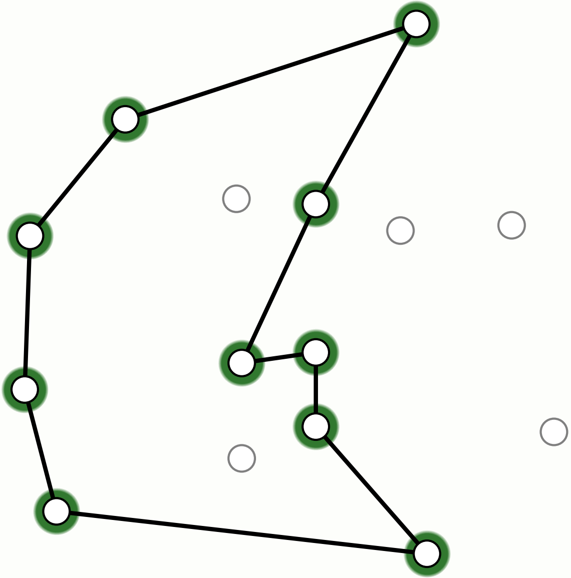
Wolsey's analysis:

$$c(T) \leq c^\top x^* \quad \text{and} \quad c(J) \leq \frac{1}{2} c^\top x^*$$

$\Longrightarrow$ LP-relative $^3/_2$-approximation



Analyzing threshold rounding: Linking the LPs

(x, y) feasible for PCTSP LP

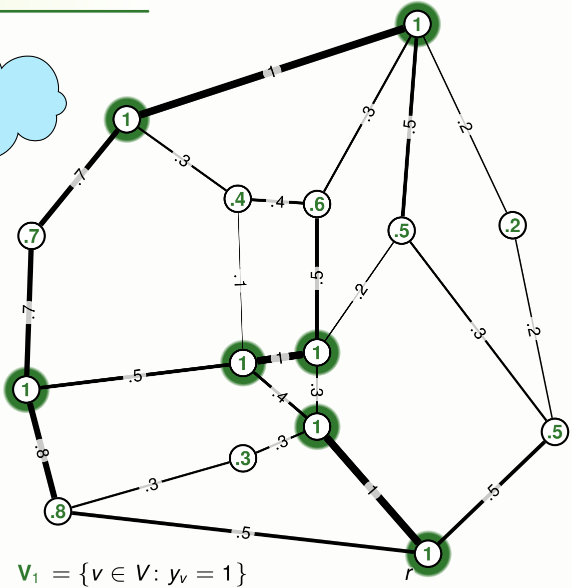
↓

z feasible for Held-Karp LP on V_1

↓

tour length $\leq \frac{3}{2} c^\top z$

$\gamma = 1$



$\mathbf{V}_1 = \{v \in V\colon y_v = 1\}$

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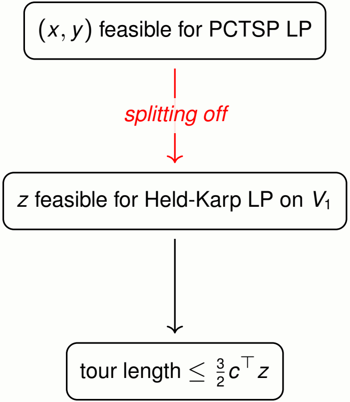

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splitting off

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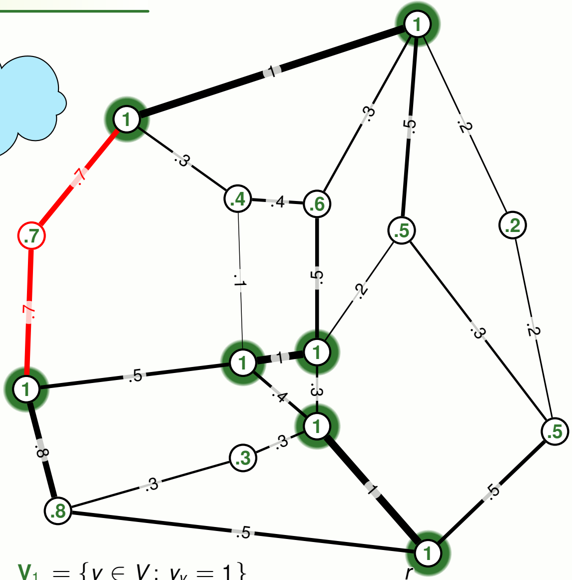

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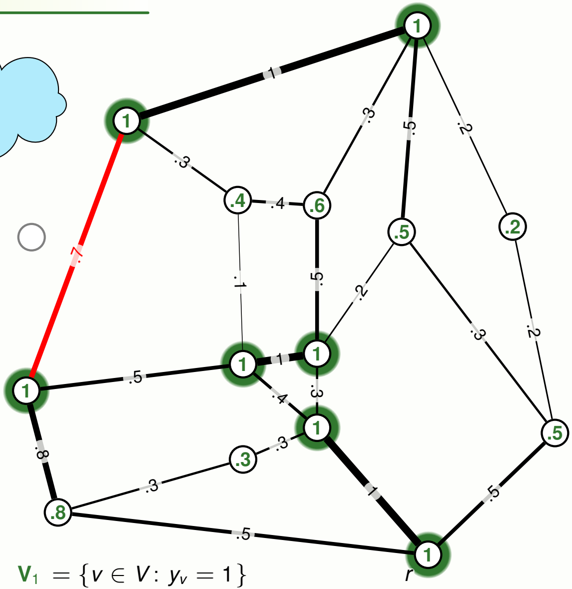

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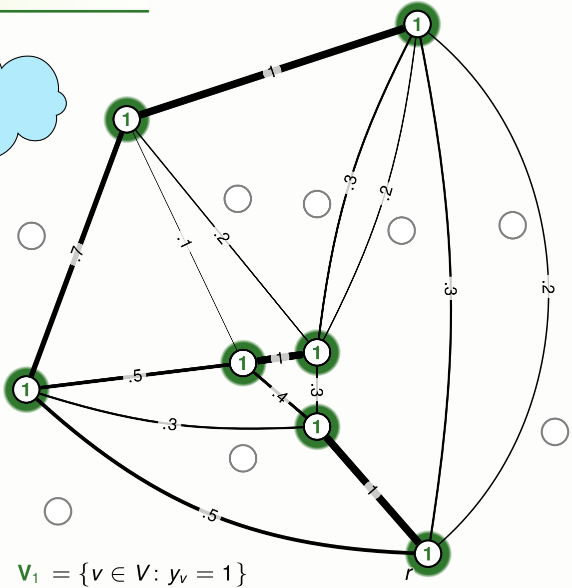

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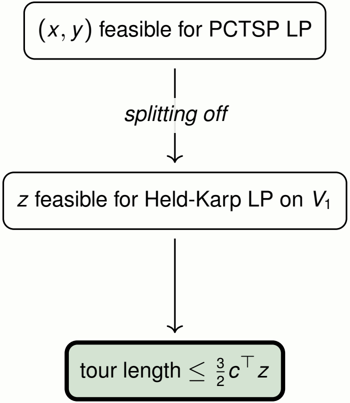

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splitting off

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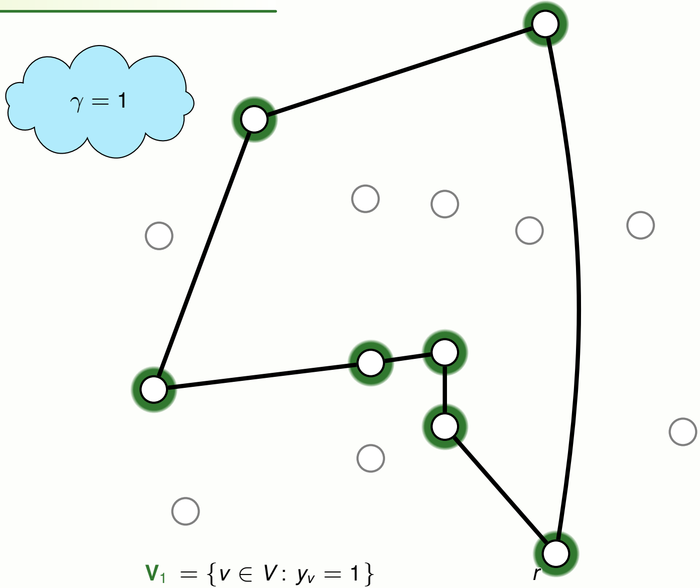

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Analyzing threshold rounding: Linking the LPs

(x, y) feasible for PCTSP LP

↓ *splitting off*

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↓

tour length $\leq \frac{3}{2} c^\top x$

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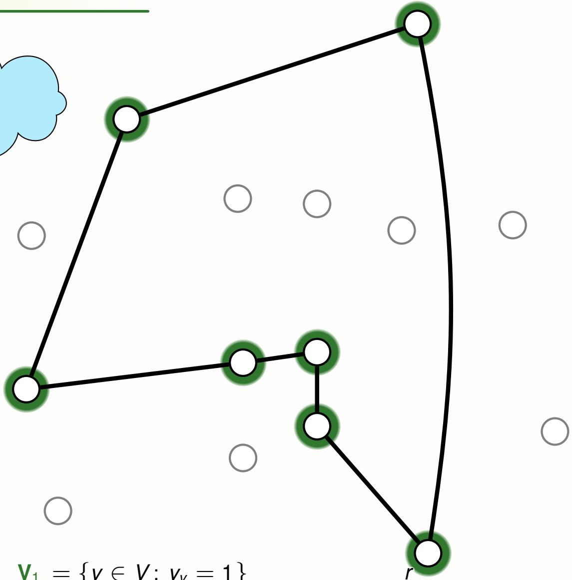


$\mathbf{V_1} = \{v \in V : y_v = 1\}$

Analyzing threshold rounding: Linking the LPs

(x, y) feasible for PCTSP LP

↓

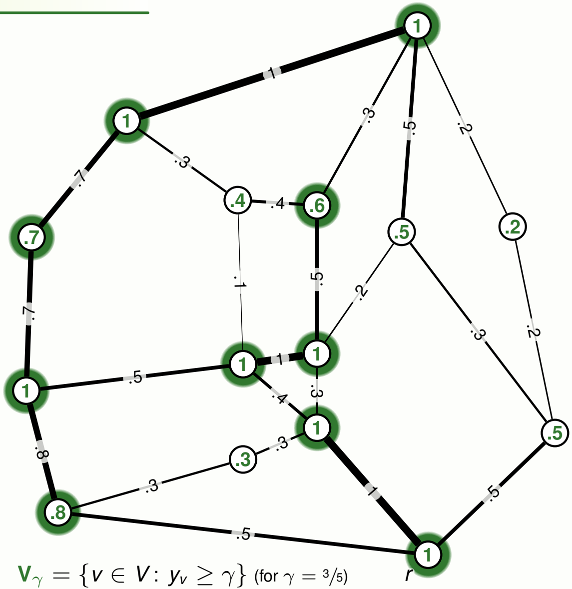
$\frac{1}{\gamma} x$ connects V_γ

↓ *splitting off*

z feasible for Held-Karp LP on V_γ

↓

tour length $\leq \frac{3}{2} \cdot c^\top z$



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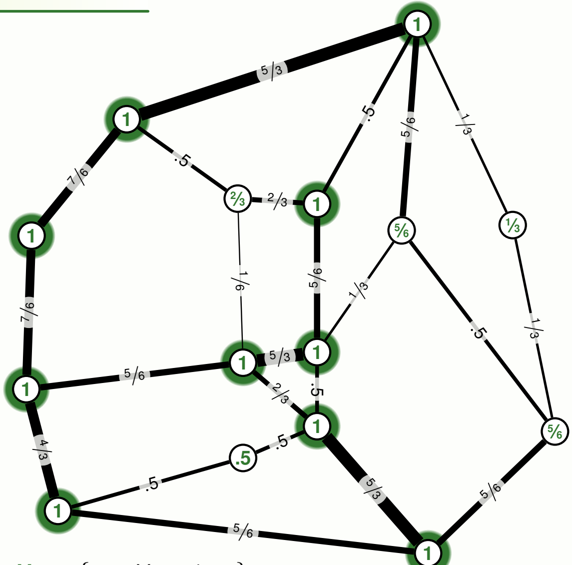

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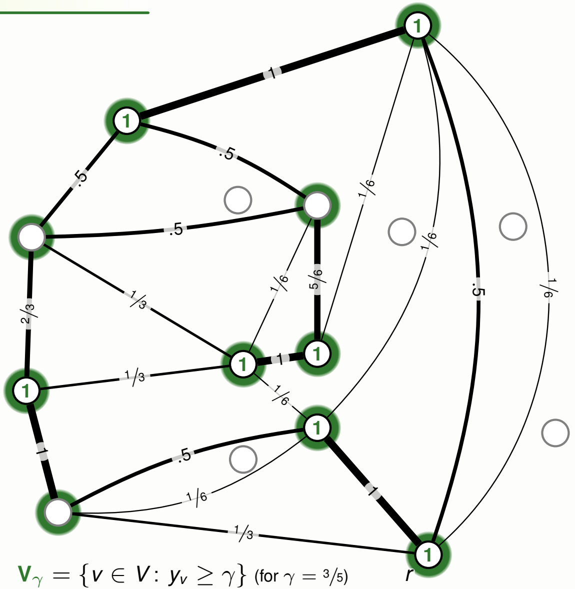

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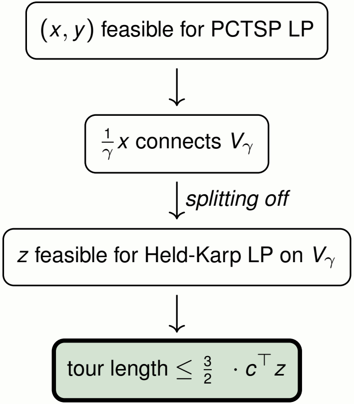

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splitting off

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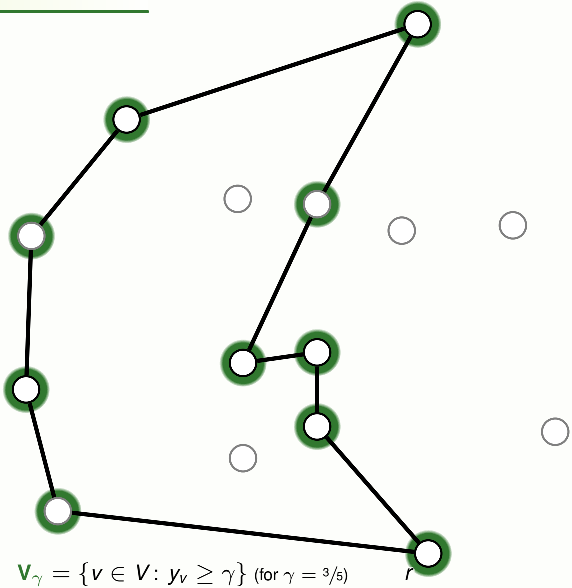

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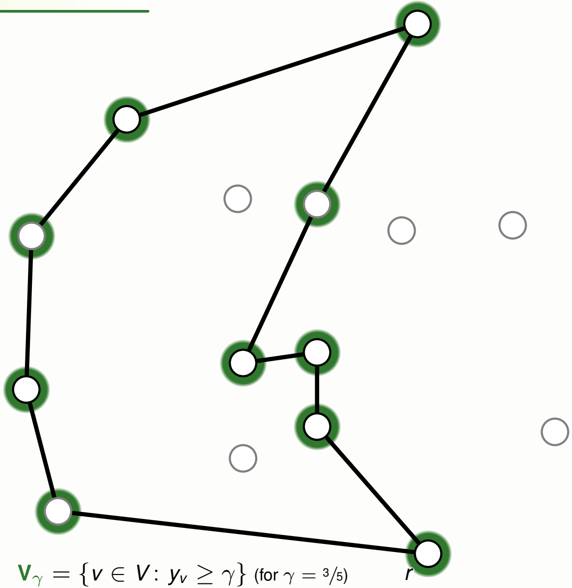

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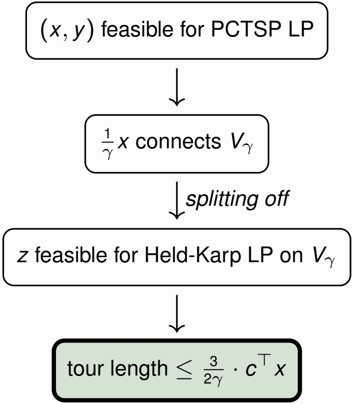

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splitting off

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Incurred penalties:

$$\sum_{v \notin V_\gamma} \pi_v = \sum_{v \in V:\ y_v < \gamma} \pi_v$$

$$\leq \sum_{v \in V:\ y_v < \gamma} \frac{1 - y_v}{1 - \gamma} \pi_v$$

Analyzing threshold rounding: Linking the LPs

(x, y) feasible for PCTSP LP

↓

$\frac{1}{\gamma} x$ connects V_γ

↓ *splitting off*

z feasible for Held-Karp LP on V_γ

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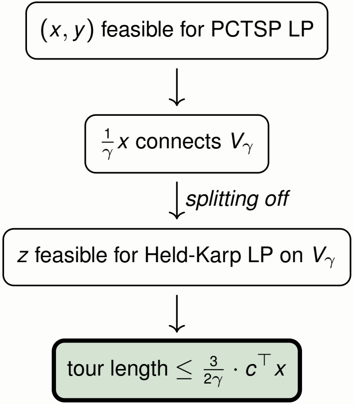

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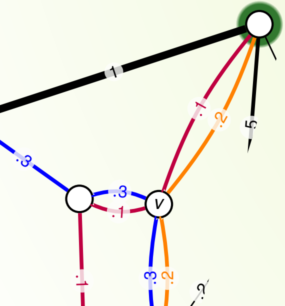
Theorem

$^5/_2$-approximation for $\gamma = {}^3/_5$.

[Bienstock, Goemans, Simchi-Levi, Williamson, 1993]

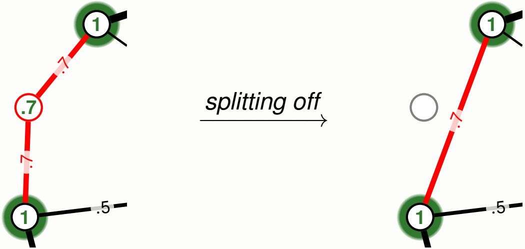
Improving the approximation guarantee

— beyond thresholding —



A major weakness of thresholding

- Transformation from LP solution x to Held-Karp solution z:



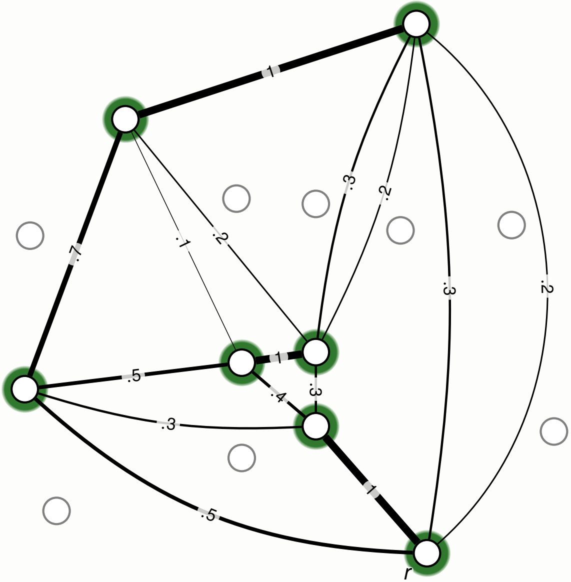
Idea: Unpack splitting off operations

- Can we stay within the budget $\frac{3}{2}c^\top x$?
- Can we improve the bound on the penalty?

Opening the blackbox

Christofides' Algorithm

1. Take a shortest spanning tree T.
2. Add a shortest $\mathrm{odd}(T)$-join J.
3. Shortcut.



$z \in P_{\mathrm{HK}}$ obtained from PCTSP solution x

Opening the blackbox

1.' Sample T with marginals $\approx z$, undo splitting off for sampled edges.

$$\implies \mathbb{E}[c(T)] \leq c^\top x$$

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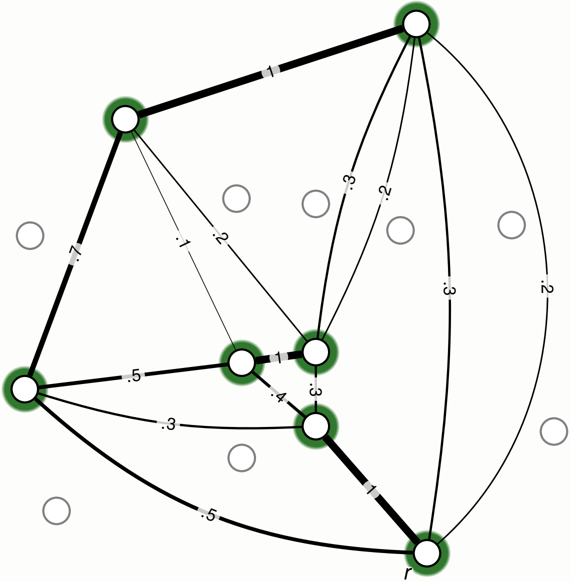

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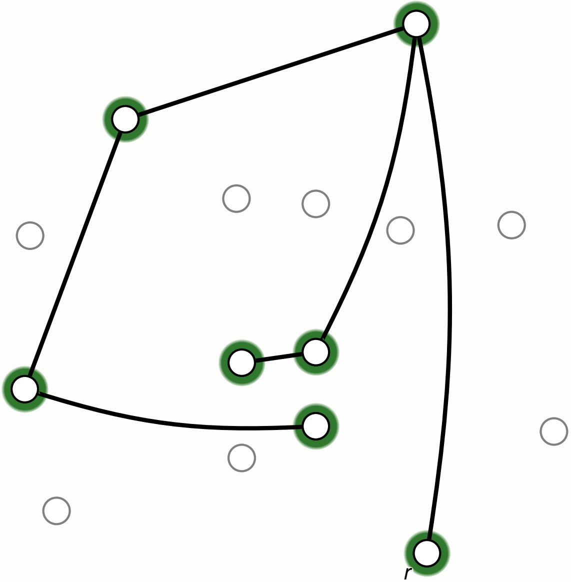

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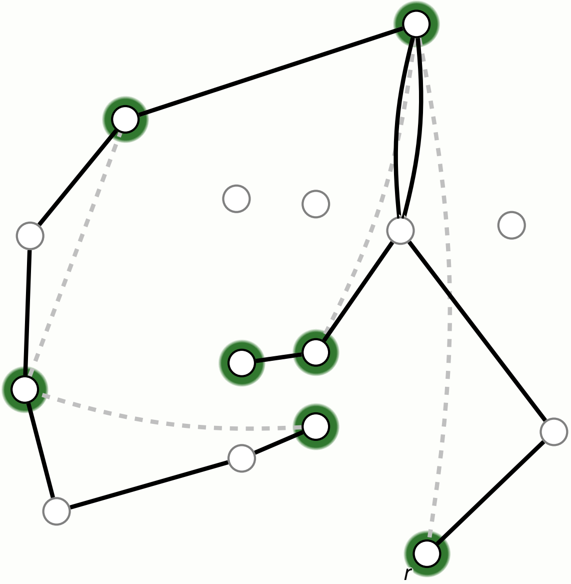

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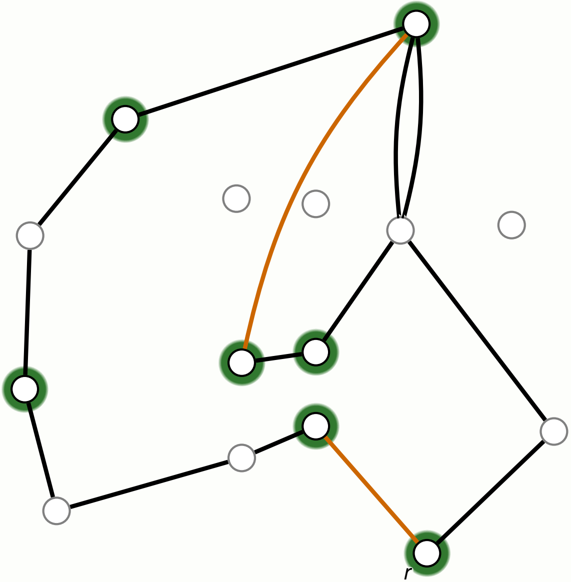

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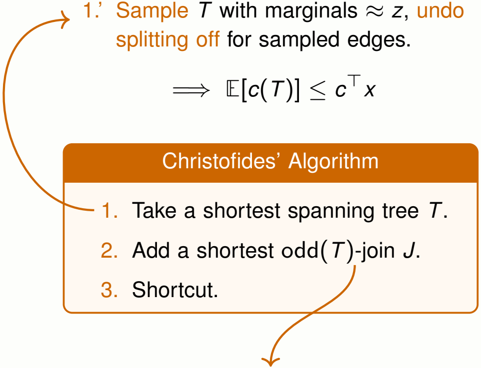

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We have $\mathrm{odd}(T) \subseteq V_\gamma$!

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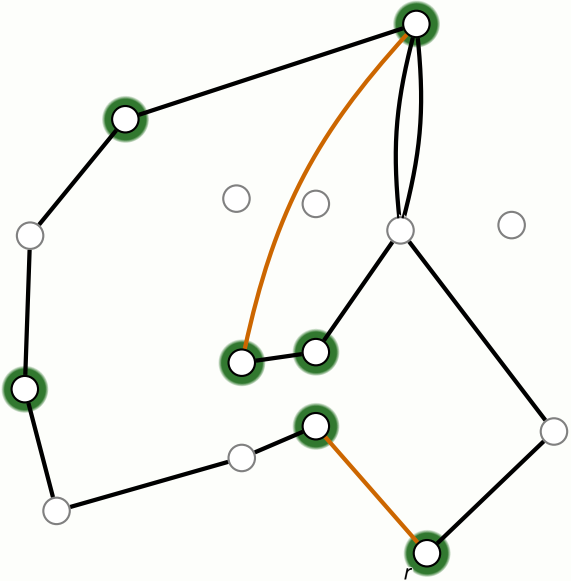

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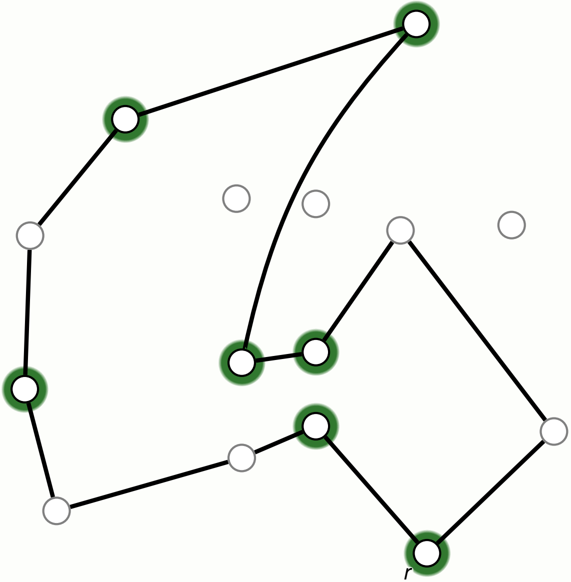
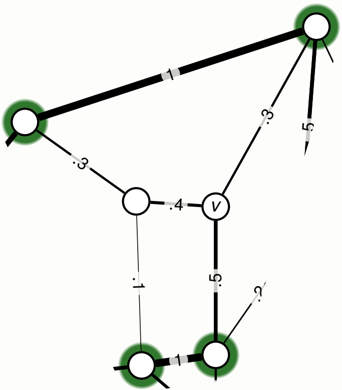
$$\implies \mathbb{E}[c(J)] \leq \tfrac{1}{2}c^\top z \leq \tfrac{1}{2}c^\top x$$

$\implies$ Extra coverage at expected cost $\frac{3}{2}c^\top x$.



Analyzing penalties

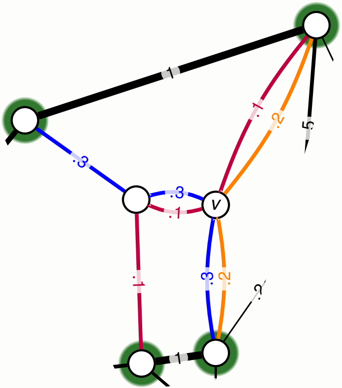


Excerpt of the PCTSP LP solution x.

For $v \notin V_\gamma$, we want to bound

$$\mathbb{P}[v \text{ is not covered by the extended tree}]$$

Analyzing penalties



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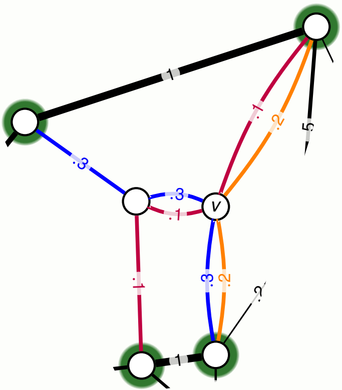

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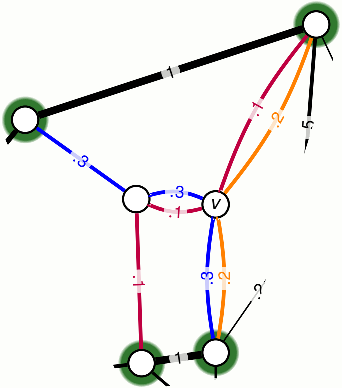
negative correlation

Analyzing penalties



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&\quad \approx \prod_{w \in W_v} (1 - z_w)
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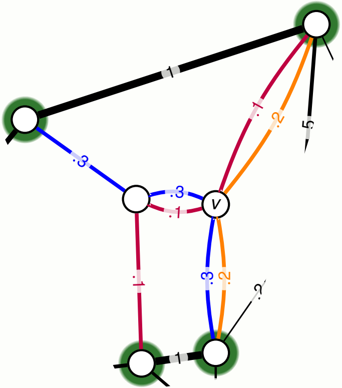

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&\quad \leq \exp\Big(-\sum_{w \in W_v} z_w\Big)
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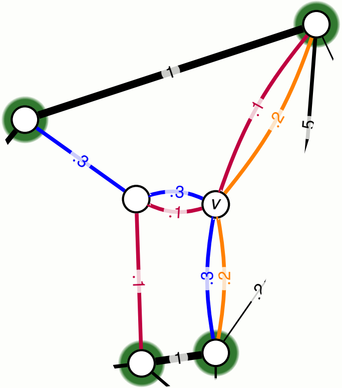
Analyzing penalties



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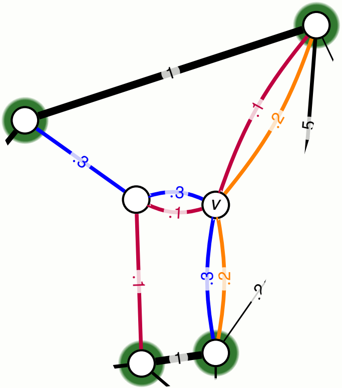
$$\approx \prod_{w \in W_v} (1 - z_w)$$

if walks pass through v once

$$\leq \exp\Bigl(- \sum_{w \in W_v} z_w\Bigr) = \exp(-y_v) \quad .$$

Analyzing penalties



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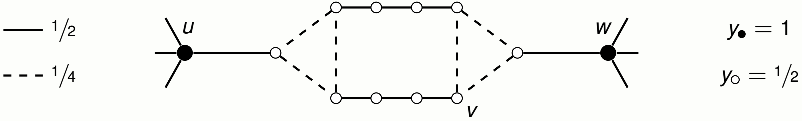
$$\le \exp\Big(- \sum_{w \in W_v} z_w\Big) = \exp(-y_v) \quad .$$

$$\Longrightarrow \text{ total expected penalty } \sum_{v \in V:\ y_v < \gamma} \exp(-y_v) \cdot \pi_v$$

We are not done yet!

(Wrong) assumption

Total weight of walks at v after undoing splitting $= y_v$.



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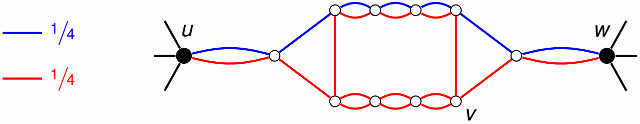

$y_\bullet = 1$

$y_\circ = 1/2$

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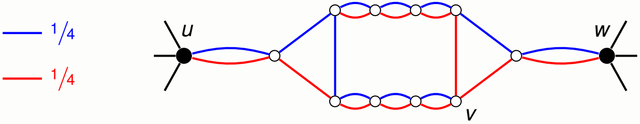

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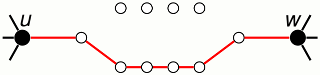
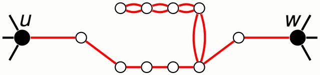
Solution

Construct trees when undoing splitting off s.t.
$y_v =$ total tree weight at v.

use the walks



and



for replacement

Wrapping up

Theorem

Let (x^*, y^*) be a PCTSP LP solution. For threshold $\gamma \in (0, 1]$, we get in poly time a cycle $C = (V_C, E_C)$ with

$$c(E_C) + \pi(V \setminus V_C) \leq \frac{3}{2\gamma} \cdot c^\top x^* + \sum_{v \in V:\ y_v^* < \gamma} \pi_v \cdot \exp\left(-\frac{3y_v^*}{4\gamma}\right) .$$

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- To compare to the LP value:

$$\exp\left(-\frac{3y_v^*}{4\gamma}\right) \leq \frac{\exp(-3/4)}{1-\gamma} \cdot (1 - y_v^*)$$

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- Better guarantee through randomized choice of γ.

A randomized threshold choice

Discretization: $\gamma_i = \frac{i}{N}$ for $i \in [N]$ and $y_j = \frac{j}{N}$ for $j \in [N]$.

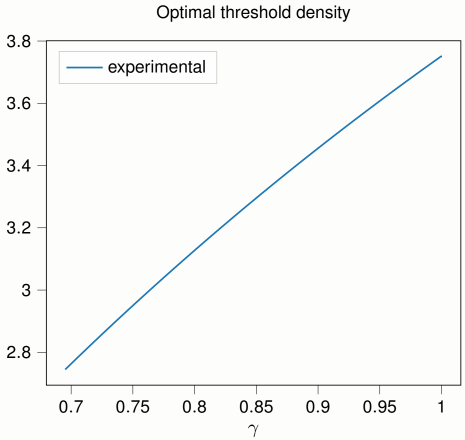
cycle cost

penalty costs at y_j

$$\min \left\{ \alpha : \begin{array}{rcll} \displaystyle\sum_{i \in \{1,\ldots,N\}} \frac{3}{2\gamma_i} z_i & \leq & \alpha & \\ \displaystyle\sum_{i \in \{1,\ldots,N\} :\, \gamma_i > y_j} \exp\left(-\frac{3y_j}{4\gamma_i}\right) z_i & \leq & (1 - y_j) \cdot \alpha & \forall j \in [N] \\ \displaystyle\sum_{i \in \{1,\ldots,N\}} z_i & = & 1 & \\ z_i & \geq & 0 & \forall i \in [N] \end{array} \right\}$$

probability for choosing threshold γ_i

A randomized threshold choice

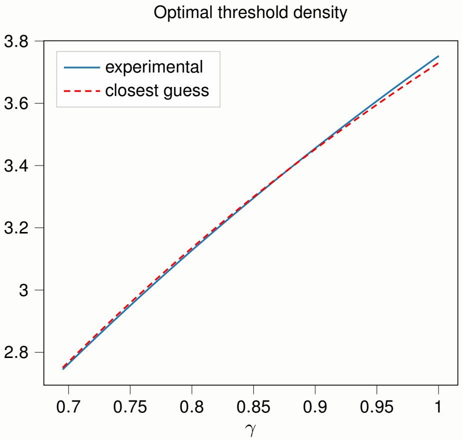


LP solution:

optimal value $\alpha \approx 1.773$

non-zero on interval $[b, 1]$ for $b \approx 0.6952$

A randomized threshold choice



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Closest guess:

Proportional to $\exp(-b/\gamma)$ on $[b, 1]$.

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- Such randomized choice of γ gives an approximation guarantee

$$\alpha = \frac{\max\left\{\int_b^1 \frac{3}{2\gamma} \exp(-{}^b\!/\!_\gamma)\, \mathrm{d}\gamma,\ \max_{y \in [0,1)} \frac{1}{1-y} \int_{\max\{b,y\}}^1 \exp\left(-\frac{3y}{4\gamma}\right) \exp(-{}^b\!/\!_\gamma)\, \mathrm{d}\gamma\right\}}{\int_b^1 \exp(-{}^b\!/\!_\gamma)\, \mathrm{d}\gamma} \ .$$

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For $b = 0.6945$, we can show $\alpha < 1.774$.

- LP technique gives computational lower bound > 1.773.

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Paper available at
arxiv.org/abs/2212.03776!