A NEW CONTRACTION TECHNIQUE WITH APPLICATIONS TO CONGRUENCY-CONSTRAINED CUTS

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PROBLEM & RESULTS

Motivation

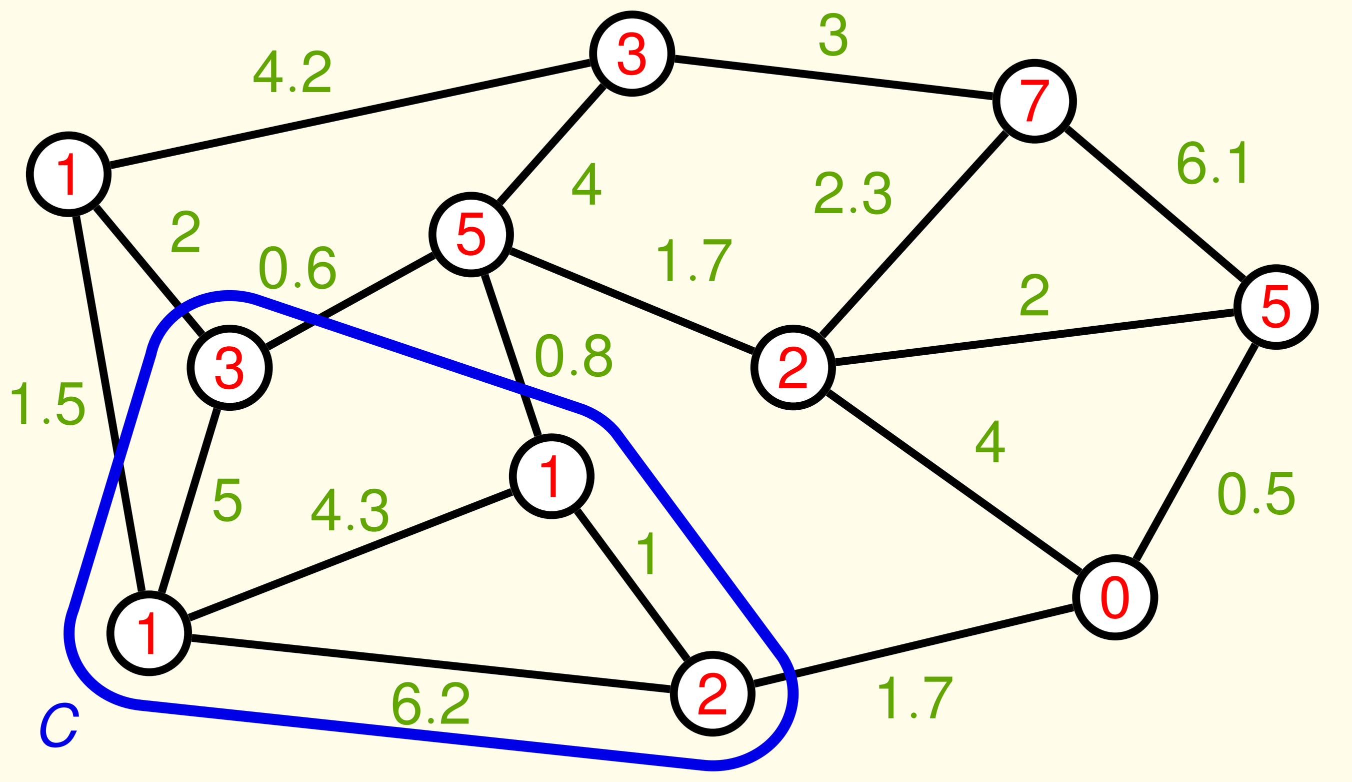
- Generalizes known problems:
  - Global minimum cut.
  - s-t cut.
  - Odd cut, even cut.
- Bounded subdeterminant IPs
  - Reduction known for bound $m = 2$.
- Submodular minimization with congruency constraints
  - Exact derministic algorithm for prime power moduli known.
  - Substantial barriers beyond.

Congruency-Constrained Minimum Cut Problem (CCMC)

Input: Graph $G = (V, E)$, weights $w\colon E \to \mathbb{R}_{\geqslant 0}$, multiplicities $\gamma\colon V \to \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{\geqslant 0}$.

Goal: Find a minimizer of

$$\min \left\{ w(\delta(C)) \;\middle|\; \begin{array}{c} \emptyset \subsetneq C \subsetneq V, \\ \sum_{v\in C} \gamma(v) \equiv r \pmod{m} \end{array} \right\}.$$



CCMC instance with opt. solution C for $m = 5$, $r = 2$.

Results

Theorem 1: PRAS for CCMC

CCMC with m constant admits a PRAS.

Theorem 2: Special case

CCMC with $m = pq$ for $p \neq q$ prime admits an exact randomized algorithm.

Theorem 3: Some Structure

CCMC with m prime and OPT $\neq 0$ admits a randomized algorithm returning poly. many s-t cut problems so that w.h.p., a cut C solves the input problem with value $\leqslant \kappa \cdot \mathrm{OPT}$ iff it solves one of the output problems with value $\leqslant \kappa \cdot \mathrm{OPT}$.

ALGORITHM & ANALYSIS

Karger's Algorithm

for unconstrained cuts

while $|V| > 2$:
Contract random edge.

Key: Singleton feasibility

$\Downarrow$

$w(\delta(v)) \geqslant \mathrm{OPT}$

$\Downarrow$

Contraction succeeds with probability $\geqslant 2/|V|$.

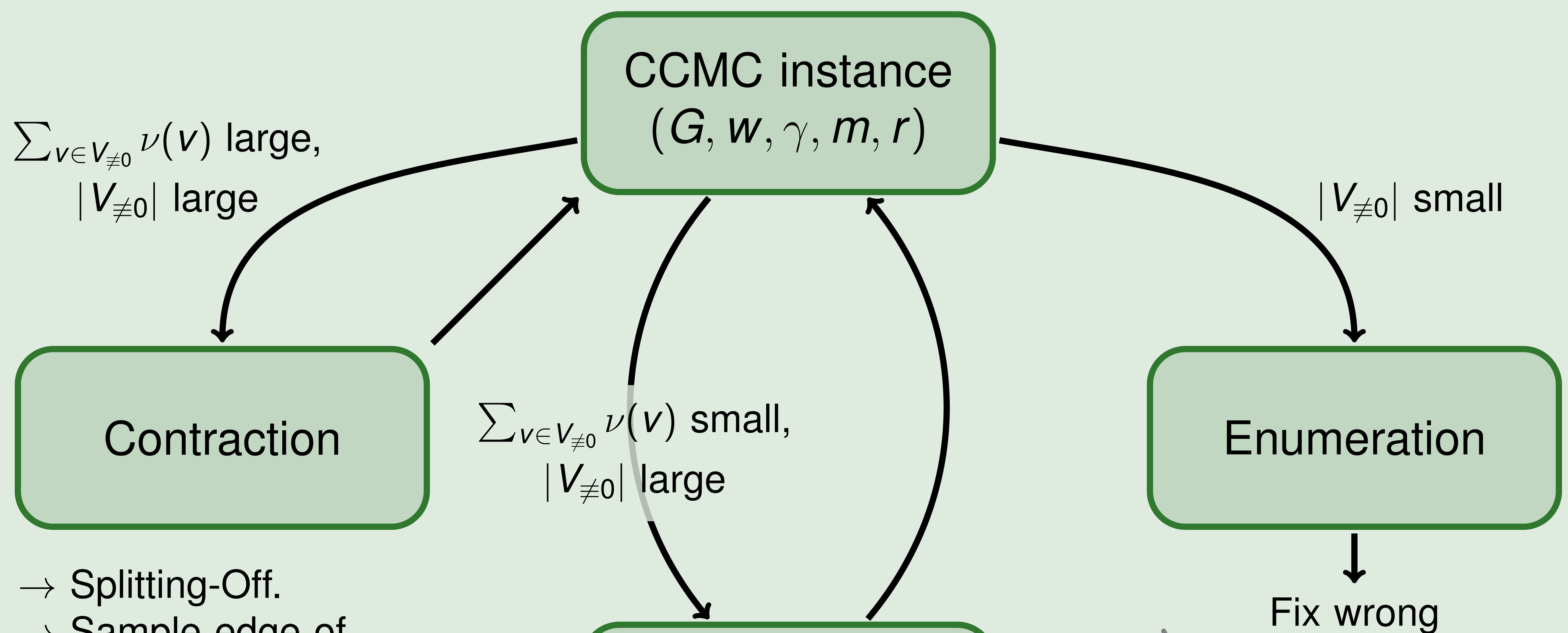
Observation

Relevant for constraint:

$V_{\neq 0} \coloneqq \{v \in V \mid \gamma(v) \neq 0\}$

Suffices to find: $\mathrm{OPT} \cap V_{\neq 0}$.

Contraction-Reduction Algorithm



Contraction:
→ Splitting-Off.
→ Sample edge of auxiliary graph.

Reduction:
→ Reduce modulus.
→ Save small cuts to fix residues.

Enumeration:
Fix wrong residues $\gamma(v)$.
Return smallest cut found.

Contraction or Reduction?

Theorem: Contraction

$$\sum_{v\in V_{\neq 0}} \nu(v) > \frac{2\mathrm{OPT}}{c} \cdot |V_{\neq 0}|$$

$\Downarrow$

Edge contraction in aux. graph succeeds with prob. $\geqslant c/|V_{\neq 0}|$.

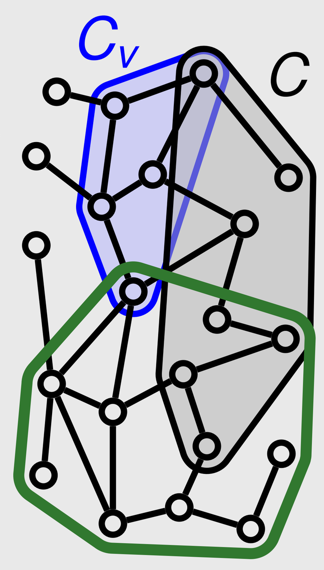
Else: $\sum_{v\in V_{\neq 0}} \nu(v)$ small and $|V_{\neq 0}|$ large.

$\Downarrow$

Many $v \in V_{\neq 0}$ with $\gamma(v) \equiv q \pmod{m}$ and $\nu(v) = w(\delta(C_v)) < \varepsilon \cdot \mathrm{OPT}$.

$\Downarrow$

Reduction to modulus $\gcd(m, q)$:

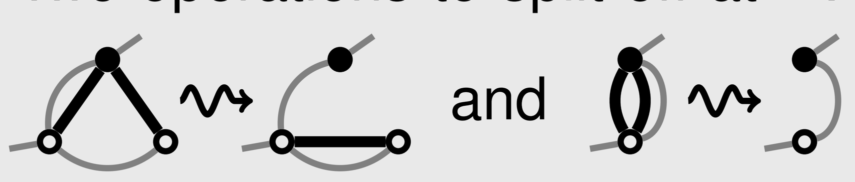


$w(C \,\Delta\, C_v) < w(\delta(C)) + \varepsilon\,\mathrm{OPT}$

$\gamma(C \,\Delta\, C_v) \equiv \gamma(C) \pm q \pmod{m}$

Splitting-Off

Two operations to split off at • :



Theorem: Efficient splitting

Can efficiently get $H = (V_{\neq 0}, F)$ with weights $w_H\colon F \to \mathbb{R}_{\geqslant 0}$ s.th.

- $\nu(v) = w_H(\delta_H(v))\ \forall v \in V_{\neq 0}$,
- cut values do not increase.

$\nu(v) \coloneqq \min\{w(\delta(C)) \mid C \cap V_{\neq 0} = \{v\}\}$

Auxiliary Graph Construction

$\nu(v) = w(\delta_G(C_v)) = w_H(\delta_H(v))$

$\gamma(C_v) = \gamma(v)$