Challenges in Congruency-Constrained TU Optimization

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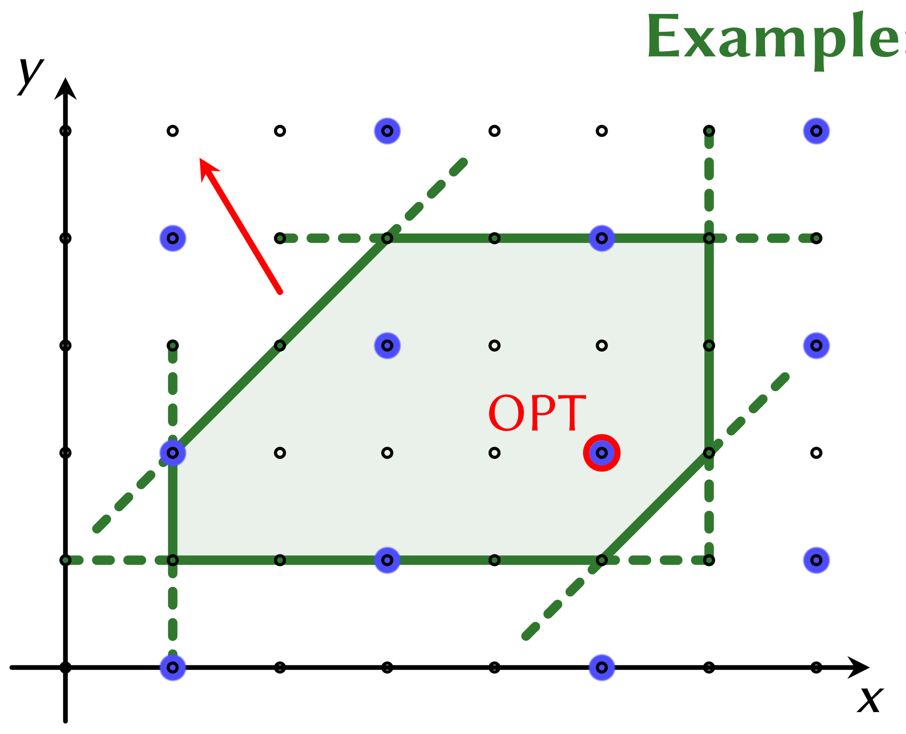
Congruency-Constrained TU Optimization (CCTU)

For $T \in \{-1,0,1\}^{k\times n}$ totally unimodular, $b \in \mathbb{Z}^k$, $\gamma \in \mathbb{Z}^n$, modulus $m \in \mathbb{Z}_{>0}$, $r \in \{0,\dots,m-1\}$, and $c \in \mathbb{Z}^n$, solve

$$\min\left\{c^\top x\colon\ Tx \le b,\ \gamma^\top x \equiv r \pmod{m},\ x \in \mathbb{Z}^n\right\}.$$

Generalization: **R-CCTU** problems for $R \subseteq \{0,\dots,m-1\}$: constraint $\gamma^\top x \in R \pmod{m}$

Example:



$$\min\ -3x+5y$$
$$\begin{pmatrix}1 & -1\\ 1 & 0\\ 0 & 1\\ -1 & 1\\ -1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} \le \begin{pmatrix}4\\ 6\\ 4\\ 1\\ -1\\ -1\end{pmatrix}$$
$$x + 2y \equiv 1 \pmod{4}$$
$$x, y \in \mathbb{Z}$$

Connections to bounded subdeterminant IPs

An IP with constraint matrix $A \in \mathbb{Z}^{k\times n}$ is *m-modular* if A has full column rank and all $n\times n$ subdeterminants are in $\{-m,\dots,m\}$.

- Reduction to CCTU for $m = 2$.
- Special case: Subdets in $\{0, \pm p\}$: (for p prime)

$$\text{Hermite Normal Form} = \begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1 & 0\\ * & \cdots & & * & p\\ * & & & * & 0/\pm p\\ & & \ddots & & \vdots\\ * & & & * & 0/\pm p\end{pmatrix}$$

→ congruency-constraint through column scaling

Why do we care?

Generalization of well-studied problems

So far: Mostly only *parity constraints*, e.g.:

- Minimum even/odd cut.
- Shortest odd cycle.

What's known?

Thm 1: CCTU Feasibility

There is a strongly poly time randomized algorithm for CCTU feasibility problems with $m \le 3$.

Extends to R-CCTU problems with prime m and $|R| \ge m-2$.

Thm 2: CCTU Optimization

There is a strongly poly time algorithm for CCTU optimization if $m \le 2$.

[Artmann, Weismantel, and Z., STOC 2017]

Extends to algorithm for R-CCTU problems with arbitrary m and $|R| \ge m-1$.

Thm 3: Reductions

For prime m, R-CCTU feasibility reduces to R-CCTU *base block* feasibility if $|R| \ge m-2$.

Extends to optimization if $|R| \ge m-1$.

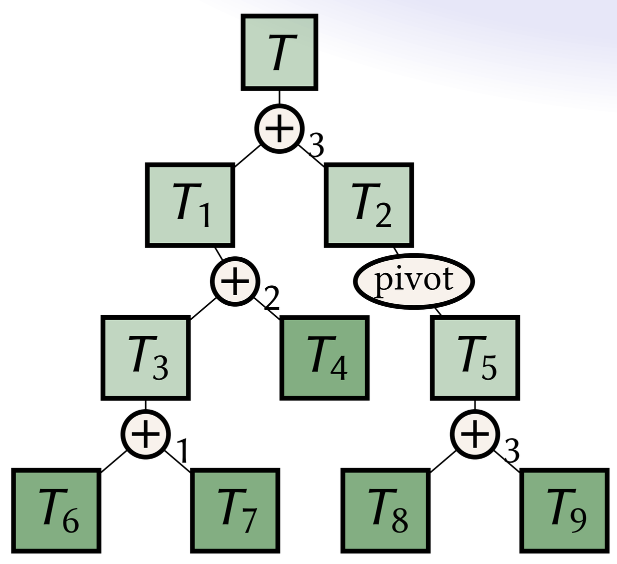
Challenges

"Divide & Conquer" through Seymour's decomposition: How to do clean integration?

Concretely, 3-sums $T = \begin{pmatrix}A & ef^\top\\ gh^\top & B\end{pmatrix}$ give subproblems

$$\begin{aligned}A x_A &\le b_A - \alpha e\\ h^\top x_A &= \beta\\ \gamma_A^\top x_A &\equiv r_A \pmod{m}\end{aligned} \qquad\text{and}\qquad \begin{aligned}B x_B &\le b_B - \beta g\\ f^\top x_B &= \alpha\\ \gamma_B^\top x_B &\equiv r_B \pmod{m}\end{aligned}$$

Can the solution structure of the smaller subproblem be integrated to the larger one?



Seymour's decomposition: Decomposing TU matrices into *base blocks*

Base block instances

Translate to combinatorial problems

Congruency-constrained cuts

Given a digraph $G = (V, A)$, find a *cut* $C \subseteq V$ minimizing $|\delta^+(C)|$ among all cuts with $|C| \equiv r \pmod{m}$.

Can we efficiently solve problems with constant but non-prime-power m?

Connections to bounded subdeterminant IPs

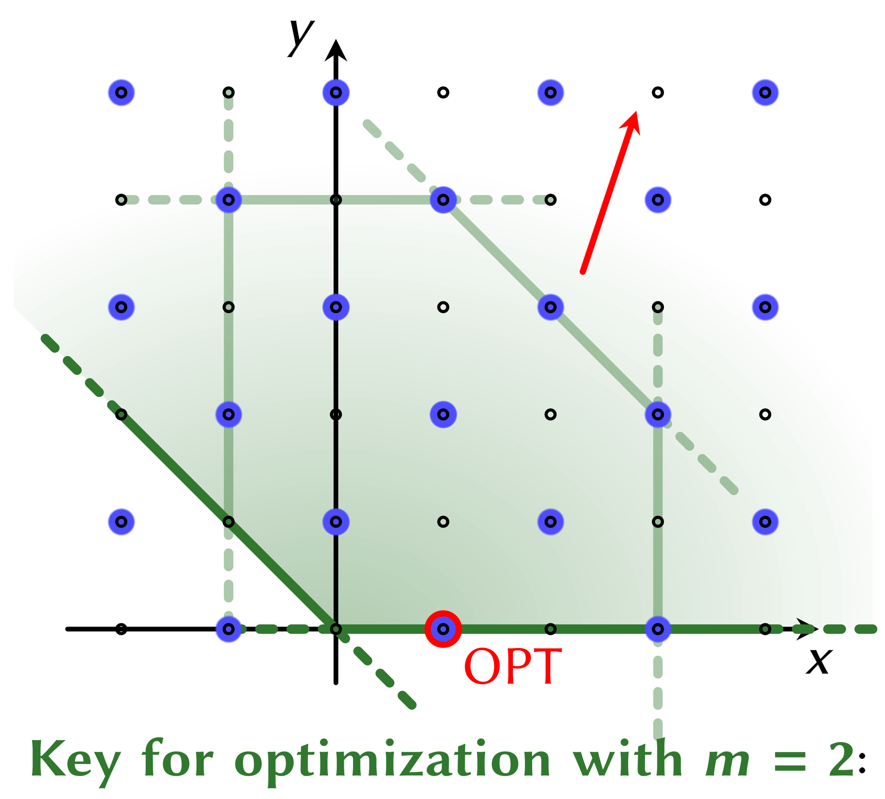
There is little knowledge on general m-modular IPs.

Can we better relate m-modular IPs and CCTU problems?

CCTU Optimization

Recent CCTU progress is on feasibility problems.

Can we push the optimization counterparts?



Key for optimization with $m = 2$: Reduction to *conic* instances

Congruency-constrained circulations

Given a digraph $D = (V, A)$, find a shortest non-zero *circulation* $C \subseteq A$ with $|C| \equiv r \pmod{m}$.

Is there an efficient deterministic algorithm?

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