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Wing-Triangulated Graphs are Perfect

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Abstract. The *wing-graph* $W(G)$ of a graph G has all edges of G as its vertices; two edges of G are adjacent in $W(G)$ if they are the nonincident edges (called *wings*) of an induced path on four vertices in G. Hoàng conjectured that if $W(G)$ has no induced cycle of odd length at least five, then G is perfect. As a partial result towards Hoàng's conjecture we prove that if $W(G)$ is triangulated, then G is perfect.

1 Introduction

A graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H equals the largest number of pairwise adjacent vertices in H. Clearly, the chordless cycles of odd length at least five (called *odd holes*) are imperfect and so are their complements (called *odd antiholes*). Graphs not containing odd holes and odd antiholes are called *Berge*. The Strong Perfect Graph Conjecture (SPGC) states that all Berge graphs are perfect. This conjecture was posed by Berge [1] in 1960 and is still open.

A way to make progress in attacking the SPGC is to prove that all graphs in some special class of Berge graphs are perfect. A classical example is the class of *triangulated graphs*, namely graphs not containing a chordless cycle of length at least four. In [1] it was shown that all triangulated graphs are perfect. The class of triangulated graphs was extended to the class of *weakly triangulated* graphs by Hayward [2]. A graph is

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weakly triangulated if it contains neither a chordless cycle of length at least five nor the complement of such a cycle. Hayward proved that all weakly triangulated graphs are perfect.

Another class of Berge graphs for which the SPGC is known to be true is the class of *strict quasi-parity graphs.* A graph is called strict quasi-parity if each noncomplete induced subgraph contains an *even pair*, namely a pair of vertices such that each induced path between these two vertices has even length. Strict quasi-parity graphs were introduced by Meyniel [5]; he proved that they are perfect.

Several classes of perfect graphs are properly contained in the class of strict quasi-parity graphs. Particularly, Hayward, Hoàng and Maffray [3] proved that all weakly triangulated graphs are strict quasi-parity.

In creating special classes of Berge graphs, Hoàng [4] suggested considering the class of wing-Berge graphs which is defined as follows. Given any graph G, construct the *wing graph* $W(G)$ by letting the vertices of $W(G)$ be the edges of G; two edges of G are adjacent in $W(G)$ if they are the *wings* of some induced path on four vertices in G, namely if they are the nonincident edges of that path. A graph G is called *wing-Berge*[4] if $W(G)$ contains no odd hole. Wing-Berge graphs are Berge (see Observation 1), and Hoàng conjectured that all wing-Berge graphs are perfect. See [4] for more information.

We call a graph *wing-triangulated* if its wing graph is triangulated. In this note we show that the conjecture of Hoàng is true for the case that the wing graph is triangulated. This was proved by the third author in her Diploma-Thesis [6]; here we will present a short proof of this result. More precisely, we shall prove a stronger result: A wing-triangulated graph is weakly triangulated or contains an even pair. In particular, all wing-triangulated graphs are strict quasi-parity.

All graphs considered here are finite, undirected and have no loops or multiple edges. We denote by P_n (resp. C_n) a path (resp. cycle) on n vertices. For a graph G, the set $N(x)$ is the *neighborhood* of the vertex x, namely the set of all vertices in G that are adjacent to x. The restriction of $N(x)$ to some induced subgraph H of G, namely the set of all vertices in H, that are adjacent to x, is denoted by $N_H(x)$. If two vertices x and y in a graph are adjacent we also say that *x sees y*; otherwise we say that *x misses y*. For an induced subgraph H we say that a vertex x is *H-partial* if it neither sees nor misses all vertices of H. A *domino* is the graph with vertex set a, b, c, d, e, f

[4] Hoàng [4] originally called these graphs *wing-perfect*, but wing-Berge seems to be a more appropriate notion for this class of graphs. Moreover, in the definition of Hoàng, the wing-graph of G has only those edges of G as vertices that are wings in at least one P_4 of G. This differs from our definition in the existence of some isolated vertices, which are unimportant with respect to perfectness.

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and edge set $ab, bc, cd, de, ef, fa, be$. The complement of a graph G is denoted by $\overline{G}$.

2 Main Theorem

By definition of a wing graph the following observations are immediate:

Observation 1

i) *if H is an induced subgraph of G, then $W(H)$ is an induced subgraph of $W(G)$.*

ii) $W(C_{2k+1}) \cong C_{2k+1}, W(C_{2k+2}) \cong 2C_{k+1}$ *for all integers* $k \geq 2$.

iii) $C_k \subseteq W(\overline{C}_k)$ *for all integers* $k \geq 5$.

iv) *no wing-triangulated graph contains a domino as an induced subgraph*

v) *no wing-triangulated graph contains a* $C_k, k \geq 7$ *or* $\overline{C}_k, k \geq 5$ *as an induced subgraph.*

Proof. *i)* and *ii)* follow immediately from the definition of a wing-graph. To prove *iii)* let the vertices of C_k be labeled $0, \ldots, k-1$. Then the k P_4's in $\overline{C}_k$ are each of the form $i, i-2, i+1, i-1$ which implies *iii)*. The wing graph of a domino contains a C_6 and therefore *iv)* holds. *v)* follows immediately from *ii)* and *iii)* □

Theorem 1 *Wing-triangulated graphs are strict quasi-parity.*

Proof. Let G be a wing-triangulated graph with no even pair.

Claim 1 *G contains neither F_1 nor F_2 (see Figure 1) as an induced subgraph.*

Assume to the contrary that G is a wing-triangulated graph that contains no even pair but contains F_1 or F_2 as an induced subgraph.

Let $\{a, b, c, d, e, f, g\}$ be the vertex set for $F_i, i = 1, 2$ with edges as shown in Figure 1. All induced paths connecting the vertices c and g in F_i have length two. Therefore there must exist a vertex x such that x sees c and misses g.

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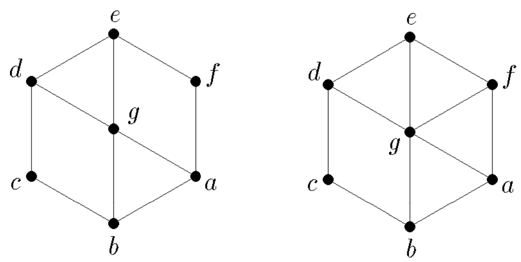

Figure 1: The graphs F_1 and F_2.

Suppose that x sees a. Then x must also see d since otherwise G contains an induced C_5. Moreover x must see e or else the edges xa, ge, cb, de induce a C_4 in $W(G)$. Now it follows that x must see b because otherwise the graph G contains an induced C_5. Considering the edges xc, ge, cb, af it follows that x must see f since otherwise $W(G)$ contains an induced C_4. But now we get a contradiction: If f misses g then the edges $cd, ga, xc, ge, cb, af, gd, fx, bg, ef$ induce a C_{10} in $W(G)$; otherwise the edges cx, ag, cd, gf induce a C_4 in $W(G)$.

Thus we have shown that x misses a and by symmetry x misses e. The vertex x cannot see f or else G contains an induced C_5 ($xfedc$ or $agdxf$) if g misses f; or if g sees f then the edges xc, gf, bc, af induce a C_4 in $W(G)$. Now x misses b since otherwise xb, ge, bc, af induce a C_4 in $W(G)$. By symmetry x also misses d. This shows that the vertex x has besides c no other neighbor on the C_6. But the edges xc, bg, fe, ba induce a C_4 in $W(G)$ or the edges xc, ba, cd, gf, bc, ed induce a C_6 in $W(G)$ depending on the existence of the edge gf, a contradiction. $\diamond$

For the following let $C = abcdef$ be an induced C_6 in G and let $P = p_0p_1 \dots p_{2k}p_{2k+1}$, $k \geq 1$ be an odd induced path between $p_0 = a$ and $p_{2k+1} = c$. The following six claims finish the proof of the theorem.

Claim 2 *The set of neighbors in C of any C-partial vertex is a subset of three consecutive vertices of C.*

Let x be a C-partial vertex that sees two opposite vertices of C, say a and d. Since G must not contain a C_5 as an induced subgraph, we may assume using symmetry that x sees also b. Now by Claim 1, x cannot see e. Thus x must see f. But then either G contains a domino or F_2 as an induced subgraph. Therefore no C-partial vertex can see two opposite vertices of C. This implies that x has at most three neighbors in C. If x

This paper appeared in: Journal of Graph Theory 24 (1997), 25-31

has exactly every second vertex of C as a neighbor then G contains a domino; otherwise the neighbors of x in C are contained in a subset of three consecutive vertices of C. ◇

Claim 3 *$p_1 \neq f$ or $p_{2k} \neq d$.*

Assume to the contrary that $p_1 = f$ and $p_{2k} = d$. The vertices b and e must each have at least one neighbor in $\{p_1, \ldots, p_k\}$ since otherwise the graph G contains an odd hole. Moreover any vertex p_i with $1 \leq i \leq 2k$ is adjacent to at most one of the two vertices b and e or else at least one of bp_iedc an bp_iefa is an induced C_5 in G. Since $p_0, \ldots, p_{2k+1}, b$ is an odd cycle in which the only possible chords start from b, it follows that the vertex b forms an odd number of triangles with the vertices $p_1, \ldots, p_{2k+1}$.

Let r and s with $0 \leq r < s \leq 2k+1$ be two indices such that e sees p_r and p_s but no vertex p_i with $r < i < s$. As b forms an odd number of triangles with P there must exist some indices r and s such that the vertex b forms an odd number of triangles with the vertices $p_r, \ldots, p_s$. Let u be the smallest index larger than r such that b is adjacent to p_u and similarly let v be the largest index smaller than s such that b is adjacent to p_v. Then $s - r$ is an even number since $ep_r \ldots p_s$ is an induced cycle of length at least five and $v - u$ is an odd number since b forms an odd number of triangles with the vertices $p_u, \ldots, p_v$. But then one of $ep_r \ldots p_u b$ and $ep_s \ldots p_v b$ is an even induced path in G and forms together with the vertices c, d or a, f an odd hole. ◇

Claim 4 *If $e \notin P$ and $p_1 = f$ then $k = 2$, $N_{P \setminus C}(d) = \emptyset, N_{P \setminus C}(b) = p_4$ and $N_{P \setminus C}(e) = p_2$.*

Since $bp_0p_1 \ldots p_{2k+1}$ is an odd cycle there must exist at least one chord of the form bp_i. Let i be the smallest value such that bp_i is an edge. Then $i = 2$ or $i = 4$ as G contains neither a C_5 nor induced cycles of length greater than six.

Assume first that $i = 2$. Then because of Claim 2 neither dp_2 nor cp_2 can be an edge. This shows that $k > 1$. Now either the edges cd, ab, cp_{2k}, bp_2 or the edges bp_{2k}, fp_2, bc, fa induce a C_4 in $W(G)$ depending on whether bp_k is an edge or not. Thus bp_2 cannot be an edge.

Now assume that bp_4 is an edge, i.e., $i = 4$. Then cp_4 must be an edge since otherwise the edges $p_3p_4, bc, af, bp_4, cd, ba$ form an induced C_6 in W(G) except when exactly one of dp_4 or dp_3 is an edge in G. But then the edges p_3p_4, bc, af, bp_4, cd form an induced C_5 in $W(G)$. At least one of the vertices d and e must see at least one of the vertices p_3 and p_2 or else the edges $de, fa, p_2p_3, fe, ab, fp_2$ induce a C_6 in $W(G)$. We will show that ep_2 is the only possible of these four edges.

This paper appeared in: Journal of Graph Theory 24 (1997), 25-31

First note that ep_4 cannot be an edge or else G contains a C_5. The vertex e cannot see p_3 since otherwise the edges ep_3, p_4b, p_2p_3, p_4c induce a C_4 in $W(G)$. If dp_2 is an edge then the edges p_2d, cp_4, de, cb show that dp_4 must be an edge or else $W(G)$ contains an induced C_4. But then the edges p_2d, bp_4, cb, ed induce a C_4 in $W(G)$. Also dp_3 cannot be an edge since otherwise G contains the odd induced cycle dp_3p_2fabc. This shows that ep_2 must be an edge. Finally note that dp_4 cannot be an edge since G must not contain an induced C_5.

Thus we have shown $k = 2, N_{P\setminus C}(d) = \emptyset, N_{P\setminus C}(b) = p_4$ and $N_{P\setminus C}(e) = p_2$. $\diamond$

Claim 5 *If $P \cap C = \{a, c\}$ then $k = 1$, $N_{P\setminus C}(e) = \emptyset$ and either $N_{P\setminus C}(f) = \emptyset$, $N_{P\setminus C}(d) = \{p_2\} \subseteq N_{P\setminus C}(b)$ or $N_{P\setminus C}(d) = \emptyset$, $N_{P\setminus C}(f) = \{p_1\} \subseteq N_{P\setminus C}(b)$*

We may assume that at least one of the two vertices d and f has at least one neighbor in $\{p_1, \ldots, p_{2k}\}$. Otherwise the odd path dPf is induced and contradicts Claim 2 applied to d and f. Using symmetry we may assume that f has at least one neighbor in $\{p_1, \ldots, p_k\}$.

Let p_f be the neighbor with largest index of f on $P - \{a, c\}$ and let p_b be the neighbor with smallest index on $P - \{a, c\}$ of b (b must have at least one such neighbor or else bP is an odd hole).

Assume first that $p_f < p_b$, i.e., p_f appears first on P while going from a to c. The only possible first neighbor p_b of b is p_2 since if bp_i is an edge for $2 \le i \le 2k$ then the edges fa, cb, ap_1, bp_i induce a C_4 in $W(G)$. Thus $p_f = p_1, p_b = p_2$ and $k = 1$ since otherwise G contains a hole. But now the edges $fa, bp_2, fp_1, p_2c, p_1a, bc$ induce a C_6 in $W(G)$, a contradiction.

Now assume that $p_f \ge p_b$. Since the edges cd, ba, cp_{2k}, bp_b must not induce a C_4 in $W(G)$ at least one of dp_b, p_bp_{2k} and bp_{2k} must be an edge.

If dp_b is an edge then Claim 2 implies that fp_b is not an edge and $p_b \ne p_1$. Then we must have $p_b = p_2$ or else the edges af, ap_1, bp_b, bc induce a C_4 in $W(G)$. But then the vertices a, b, c, d, p_1, p_b induce a domino in G. Thus dp_b cannot be an edge.

Now assume that dp_b is not an edge but p_bp_{2k} is an edge. Then we must have $p_b = p_1$ which implies $k = 1$ or else G contains a hole. By Claim 2 f is adjacent to p_1 and not adjacent to p_2. Now ep_1 is not an edge because of Claim 2 and ep_2 also is not an edge because otherwise G contains a domino on the vertices e, f, p_1, p_2, b, c or if bp_2 is an edge then Claim 2 forbids the edge ep_2. Now dp_2 cannot be an edge or else G contains a C_5. Thus we have shown $k = 1, N_{P\setminus C}(d) = \emptyset$ and $N_{P\setminus C}(f) = \{p_1\} \subseteq N_{P\setminus C}(b)$.

This paper appeared in: Journal of Graph Theory 24 (1997), 25-31

Finally assume that neither dp_b nor p_bp_{2k} is an edge but bp_{2k} is an edge. Then $p_b = p_1$ must hold or else the edges fa, bc, ap_1, bp_{2k} induce a C_4 in $W(G)$. Since fp_{2k} cannot be an edge because of Claim 2, the vertex f must also miss p_1 or else the edges fp_1, bc, af, bp_{2k} induce a C_4 in $W(G)$. Now b must see p_f or else the edges ef, ap_1, fp_f, ab induce a C_4 in $W(G)$ (note that $p_f \neq p_2$ since otherwise applying Claim 3 to the path $afp_f \dots p_{2k}c$ we get a contradiction). But then the edges ef, ap_1, fp_f, bp_1, cd induce a C_5 in $W(G)$, since neither d nor e can be adjacent to p_1 or p_f because of Claim 2. ◇

Since we assumed that G does not contain an even pair, it cannot be weakly triangulated thus by Observation 1 G must contain an induced C_6. Let $C = x_1x_2 \dots x_6$ be such an induced C_6 in G. Then there must exist an induced odd path P between x_1 and x_3. Without loss of generality we may assume that the path P does not contain the vertex x_5; otherwise consider the pairs $\{x_1, x_5\}$ or $\{x_3, x_5\}$. By claims 3, 4 and 5 and by symmetry G contains an F_3 or an F_4 or an F_5 as an induced subgraph (see Figure 2).

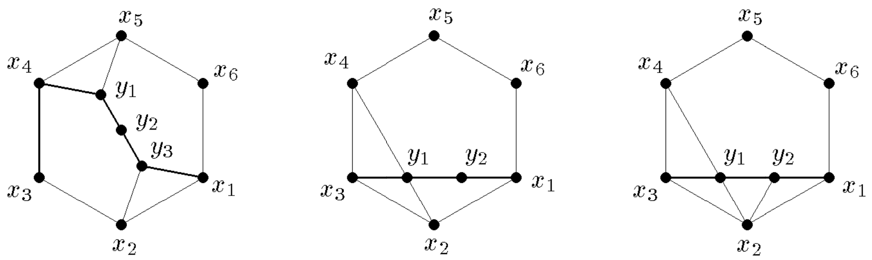

Figure 2: The graphs F_3, F_4 and F_5.

In the following two claims, we will show that G cannot contain F_3 or F_4 or F_5 as an induced subgraph and thus yield a contradiction. This finishes the proof.

Claim 6 *G cannot contain an induced F_3.*

Suppose that G contains an induced F_3.

First we will show that any vertex different from x_1 and x_5 that sees x_6 must also see $x_1, \dots, x_5$. Assume that there is a vertex z contradicting this claim. Then neither the edges $x_2x_1, x_6x_5, x_1y_3, x_6z$ nor the edges $x_1x_6, x_5x_4, x_6z, x_5y_1$ may induce a C_4 in $W(G)$. Thus z must see at least one of the vertices x_1, x_2, y_3 and at least one of the vertices x_5, x_4, y_1. If z sees x_2 or y_3 then it cannot see any of the vertices x_5, x_4, y_1 or

This paper appeared in: Journal of Graph Theory 24 (1997), 25-31

else we get a contradiction to Claim 2 with one of the three induced C_6's contained in F_3. Thus z must see x_1 and by symmetry also x_5. Claim 2 now yields that z cannot have any other neighbor in F_3. But then the edges x_5z, x_1y_3, x_6x_5 and x_1x_2 induce a C_4 in $W(G)$. Thus the vertex z cannot exist. By symmetry the same holds for the vertex x_3.

As $\{x_2, x_4\}$ is not an even pair, there exists an induced x_2, x_4-path Q of odd length. As shown above any vertex adjacent to x_6 must also see $x_1, \ldots, x_5$ which shows that x_6 cannot belong to Q. By Claim 3, at most one of x_1, x_5 belongs to Q. If none of x_1, x_5 belongs to Q then Claim 5 shows that there must exist a vertex seeing x_3 but not for example x_6, which we have shown above is not possible. Thus one of x_1, x_5 must belong to Q. Using symmetry we may assume that x_1 belongs to Q. But then Claim 4 shows that there must exist a vertex that sees x_6 but not all of $x_1, \ldots, x_5$, a contradiction as we have shown above. ◇

Claim 7 *G cannot contain an induced F_4 or F_5.*

Suppose that G contains an induced F_4 or F_5. By Claim 6 we know that G contains no induced F_3.

Again we will show that any vertex different from x_1 and x_5 that sees x_6 must also see $x_1, \ldots, x_5$. Assume that there is a vertex z contradicting this claim. Then Claim 2 implies that neither zy_1 nor zx_3 can be an edge. Since the edges $zx_6, x_1x_2, x_5x_6, x_1y_2$ must not induce a C_4 in $W(G)$ we know that at least one of zx_1, zx_2, zy_2 must be an edge. In any case Claim 2 implies that zx_4 cannot be an edge. Also zy_2 is not an edge or else $zy_2, y_1x_4, x_1y_2, y_1x_3$ induce a C_4 in $W(G)$. Similarly zx_2 is not an edge or else the edges $x_6z, x_2y_1, x_1x_6, x_2x_3$ induce a C_4 in $W(G)$. Now we can conclude that zx_1 must be an edge since above we observed that at least one of zx_1, zx_2, zy_2 is an edge. But then $zx_1, x_2y_1, x_1x_6, y_1y_2$ induce a C_4 in $W(G)$.

Since the vertices x_2, x_4 may not form an even pair in G there must exist an odd induced path between them. This path cannot contain x_6 since we just proved that any vertex adjacent to x_6 must see all of $x_1, \ldots, x_5$. By Claims 3, 4, 5 and 6 this path must be of length three and contains none of the vertices $x_3, x_1, x_5, x_6, y_1, y_2$. Thus a neighbor z of x_4 exists that does not see x_2. Then z must see x_3 or y_1 or else the edges $zx_4, x_2x_3, x_4x_5, x_2y_1$ induce a C_4 in $W(G)$.

Assume first that zx_3 is an edge. By Claim 2 the vertex z sees neither x_1 nor x_6. Moreover z cannot see y_2 or else G contains an induced C_5. Thus z must see y_1 since otherwise the edges $zx_4, y_1x_2, y_1y_2, x_1x_6$ induce a C_4 in $W(G)$. But then $zy_1, x_2x_1, y_1x_4, x_1y_2$ induce a C_4 in $W(G)$.

This paper appeared in: Journal of Graph Theory 24 (1997), 25-31

Now assume that zx_3 is not an edge and therefore zy_1 is an edge. Then zy_2 must be an edge or $zy_1, x_2x_1, y_1x_4, x_1y_2$ induce a C_4 in $W(G)$. But now Claim 2 shows that z has no neighbor in $\{x_5, x_6, x_1\}$ and therefore $W(G)$ contains an induced C_6 formed by the edges $x_5x_4, x_3x_2, x_1y_2, y_1x_4, x_1x_2, y_2z$. ◇

□

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