This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

On Wing-Perfect Graphs

Stefan Hougardy [1]

revised version, October 1997

Abstract. An edge in a graph G is called a *wing* if it is one of the two non-incident edges of an induced P_4 (a path on four vertices) in G. For a graph G its *wing-graph* $W(G)$ is defined as the graph whose vertices are the wings of G and two vertices in $W(G)$ are connected if the corresponding wings in G belong to the same P_4. We will characterize all graphs whose wing-graph is a cycle. This solves a conjecture posed by Hoàng [9].

1 Introduction

A P_4 is a path on four vertices. Two graphs G and H are called P_4-*isomorphic* if there exists a bijection between the vertices of G and H such that four vertices induce a P_4 in G if and only if their images under this bijection induce a P_4 in H. The study of P_4-isomorphic graphs was initiated in 1984 by a conjecture of Chvátal [1]. He conjectured that if a graph G is P_4-isomorphic to a perfect graph then G is perfect. In 1987 this conjecture was proved by Reed [12] and this result is considered as a major progress in trying to resolve Berge's famous Strong Perfect Graph Conjecture. (For more background on perfect graphs see [6]).

Reed's result shows that the perfectness of a graph depends solely on the structure of its P_4's. This was a motivation to find decomposition schemes for perfect graphs and classes of perfect graphs that were defined only in terms of P_4's. See [3], [8], [2], [4], [7] and [10] for such kind of results.

[1] Humboldt-Universität zu Berlin, Institut für Informatik, Lehrstuhl Algorithmen und Komplexität, Unter den Linden 6, 10099 Berlin, Germany, hougardy@informatik.hu-berlin.de

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

Another approach in deriving information from a graph G by using only the structure of its P_4's uses the notion of a *wing*. An edge in a graph G is called a *wing* if it is one of the two non-incident edges of an induced P_4 in G. For a graph G its *wing-graph* $W(G)$ is defined as the graph whose vertices are the wings of G and two vertices in $W(G)$ are connected if the corresponding wings in G belong to the same P_4.

Hoàng [9] has conjectured that a graph is perfect if its wing-graph is bipartite. As suggested by Chvátal [5], graphs whose wing-graph is bipartite are therefore called Hoàng-graphs. Up to now it is not known whether Hoàng graphs are perfect. While attacking this problem, Hoàng [9] made the following conjecture concerning wing-graphs:

Hoàng's Conjecture *If G is a graph such that every vertex belongs to a P_4 and $W(G)$ is isomorphic to an odd cycle of length at least five then G or its complement $\overline{G}$ is an odd cycle of length at least five.*

In this paper we will characterize all graphs whose wing-graph is a cycle. Thereby we prove that Hoàng's conjecture is true with the only exception of the graph F_{34} (see Figure 4) that is not an odd cycle, but whose wing-graph is a C_9.

Here is a summary of the results we will present in this paper (the graph G is assumed to have the property that every vertex belongs to some induced P_4 in G). The graphs F_1–F_{35} are shown in Figure 1–4.

- $W(G) \cong C_3 \Leftrightarrow G \cong F_1$
- $W(G) \cong C_4 \Leftrightarrow G \in \{F_{10}, \ldots, F_{21}\}$
- $W(G) \cong C_5 \Leftrightarrow G \cong C_5$
- $W(G) \cong C_6 \Leftrightarrow G \in \{\overline{C}_6, F_7, F_{31}, F_{32}, F_{33}\}$
- $W(G) \cong C_9 \Leftrightarrow G \in \{C_9, \overline{C}_9, F_{34}\}$
- $W(G) \cong C_{12} \Leftrightarrow G \in \{\overline{C}_{12}, F_{35}\}$
- $W(G) \cong C_k, k$ even and $k \notin \{4, 6, 12\} \Leftrightarrow G \cong \overline{C}_k$
- $W(G) \cong C_k, k$ odd and $k \notin \{3, 5, 9\} \Leftrightarrow G \in \{C_k, \overline{C}_k\}$

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

2 Notations

Given two vertices x and y in a graph G we say that x *sees* y if x and y are connected by an edge in G. If x does not see the vertex y then we say that x *misses* y. Given a graph G and some set S of edges of G we define the graph *induced by* S to be the graph that is induced by the end vertices of the edges in S.

A path (resp. cycle) on k vertices is denoted as P_k (resp. C_k). For a path on four vertices we often will just list its set of vertices, e.g. $abcd$ stands for the path on vertices a, b, c and d with edges ab, bc and cd. The complement of a graph G is denoted by $\overline{G}$.

If a graph G contains a vertex x that does not belong to any induced P_4 of G then the wing-graph of G is obviously isomorphic to the wing-graph of $G - x$. This observation leads to the definition of P_4-*dense* graphs. We call a graph P_4-dense if every vertex belongs to at least one induced P_4. Given an arbitrary graph G one easily can determine all vertices of G that do not belong to an induced P_4 of G. By removing this set of vertices one gets a P_4-dense subgraph of G which has the same wing-graph as G. This shows that for our characterization of all graphs whose wing-graph is a cycle it is enough to consider only P_4-dense graphs.

We denote the end of a proof by $\Box$ and the end of a proof of a claim contained within a proff by $\Diamond$.

3 Graphs whose wing-graphs are short paths or cycles

Our main theorem in Section 5 essentially states that the only graphs whose wing-graph is a cycle are the odd cycles and complements of cycles. However this is only true for long cycles. For small cycles there exist several exceptions. In this section we will deal with two such exceptions; the case of C_3 and C_4. Moreover we are presenting a list of induced subgraphs such that every graph whose wing-graph contains a P_3 must contain at least one of these graphs as an induced subgraph. This result will be extended to P_4's in the next section and is one main ingredient of the proof for our main result.

Lemma 1 *Let G be a graph such that its wing-graph $W(G)$ has maximum degree 2. If $W(G)$ contains a path on three vertices then the corresponding three edges in G induce C_5, C_6, P_6 or one of the graphs $F_1 - F_9$ (see Figure 1) in G.*

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

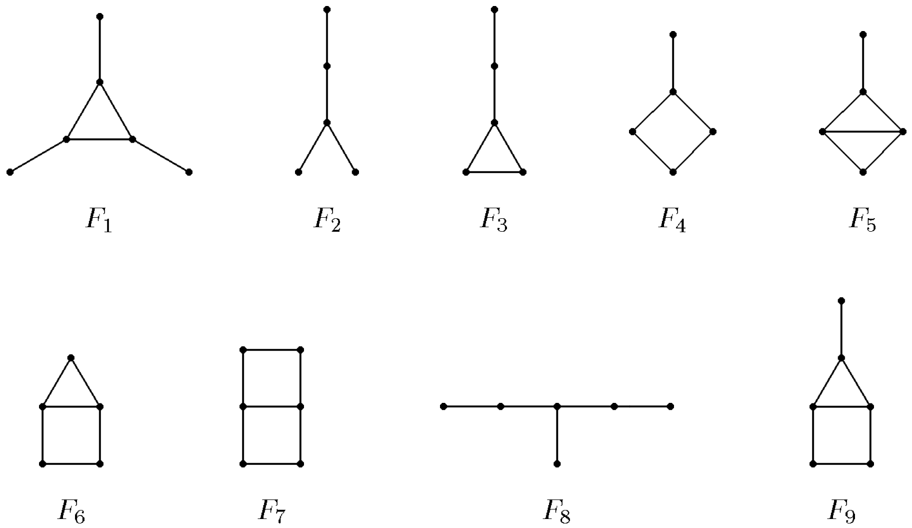


Figure 1: Graphs with a P_3 in their wing-graph.

Proof. Let ab, cd, ef be three wings in G such that $abcd$ induces a P_4 and edges ab and ef are the wings of the same P_4 in G. Let us first assume that the wing ef has some vertex in common with the P_4 $abcd$. If $c = e$ then f misses a and b and thus we get either graph F_2 or F_3. If $d = e$ then f must see exactly one of a and b. If f sees a then we get C_5 or F_6. If f sees b then we get F_4 or F_5.

Now let us assume that the wing ef does not have some vertex in common with the P_4 $abcd$. Then exactly one of the vertices e, f sees exactly one of the vertices a, b. Using symmetry we may assume that e sees a or b and f misses a and b.

If vertex e sees vertex a, then f must miss c and e must miss d or else edge ab would be a wing in three different P_4's. If f sees d then we get a C_6 or F_7. If f misses d then e must miss c or else ef would be contained in three different P_4's. Thus we get a P_6 in this case.

Next assume that e sees b. Then e must miss d and f must miss c or else edge ab would be a wing in three different P_4's. If e see c then we get F_1 or F_9. If e misses c then f must miss d as otherwise ef would belong to three different P_4's and thus we get F_8. □

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

As an immediate consequence of Lemma 1 we get a characterization of those subgraphs in G that are induced by three consecutive wings in $W(G)$. Moreover we get a characterization of all graphs whose wing-graph is a cycle of length 3 or 4.

Lemma 2 *Let G be a graph such that its wing-graph $W(G)$ is a cycle of length at least five. The graph that is induced by any 3 edges in G that are consecutive wings in $W(G)$ is C_5, P_6 or one of the graphs $F_2 - F_9$ (see Figure 1).*

Proof. It is easily verified that the wing-graphs of the graphs C_5, P_6 and $F_2 - F_9$ are either cycles of length at least five or induced subgraphs of such cycles. The wing-graph of C_6 are two disjoint triangles, the wing-graph of F_1 is a triangle, therefore these two graphs from Lemma 1 have to be omitted. □

Corollary 1 *The graph F_1 is the only P_4-dense graph whose wing-graph is a triangle.*

□

In [11] it has been shown that any graph whose wing-graph is triangulated (i.e. contains no induced cycle of length four or more) is perfect. The following corollary was a useful tool for proving this result.

Corollary 2 *The graphs F_{10}-F_{21} (see Figure 2) are the only P_4-dense graphs whose wing-graph is a C_4.*

Proof. The proof of this result is implicitly contained in the proof of Lemma 3. As this corollary is stated here just for completeness we omit the details of the proof. □

4 Graphs whose wing-graph contains a P_4

In this section we extend the result of Lemma 2 to P_4's. This will be the starting point for the proof of our main theorem in Section 5.

Lemma 3 *Let G be a graph such that its wing-graph $W(G)$ is a cycle of length at least five. The graph that is induced by any 4 edges in G that are consecutive wings in $W(G)$ is C_5, $\overline{C}_6$, C_7, F_7, P_8 or one of the graphs $F_{22} - F_{31}$ (see Figure 3).*

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

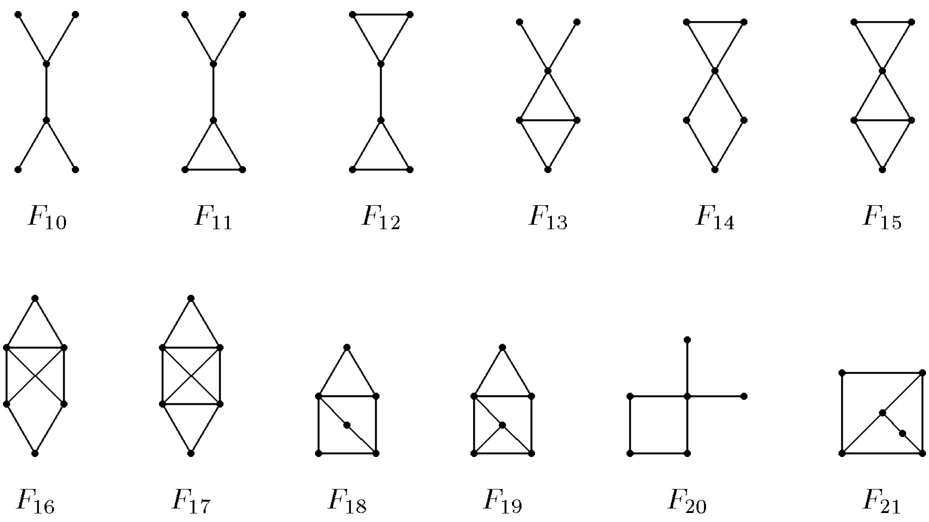


Figure 2: Graphs with a C_4 as their wing-graph.

Proof. Let w_1, w_2, w_3, w_4 be four edges in G that are consecutive wings in $W(G)$. We denote by H respectively H' the graph induced by the edges w_1, w_2, w_3 respectively w_1, w_2, w_3, w_4. By Lemma 2 we know that H is C_5, P_6 or one of the graphs $F_2 - F_9$. In case of C_5 or F_7 we are immediately done, as these two graphs have a C_5 respectively C_6 as their wing graph. If H is one of the graphs $P_6, F_2 - F_6, F_8, F_9$ note that it is uniquely determined (up to the symmetry of the graphs) which three edges are the wings w_1, w_2, w_3. This is even the case for the graphs F_8 and F_9 where there are several choices of three consecutive wings, but only one of them induces the right graph.

We now have to extend the graphs $P_6, F_2 - F_6, F_8, F_9$ each by a fourth wing to see that this extension results in the graphs stated in the lemma.

claim 1 *If $H = F_2$ then H' is isomorphic to F_{22}, F_{24} or F_{29}.*

Let the vertices of H be labeled a, b, c, d, e such that $abcd$ induces a P_4 and the vertex e is connected to c. We now have to add a wing fg to H that induces a P_4 together with edge ce to obtain the graph H'. Let us first assume that edge fg has vertex g with the graph H in common.

If $a = g$ then vertex f must see exactly one of c and e. If f sees e then it must also see b as otherwise H' contains a C_5. As edge cd must not be a wing in three different P_4's we must have the edge fd. But then $W(H')$ is a C_4. If f sees c then it must also

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

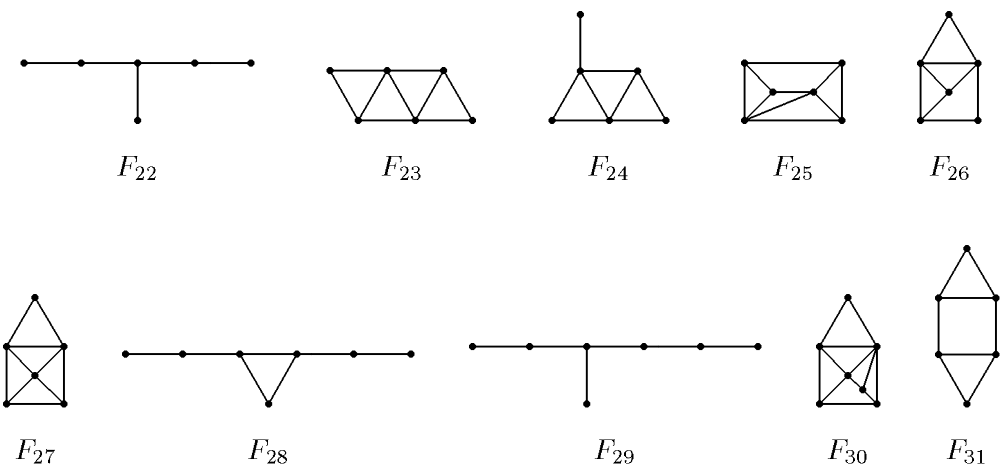


Figure 3: Graphs with a P_4 in their wing-graph.

see d as otherwise $W(H')$ contains a C_4. But then f must also see b or else edge ab is a wing in three different P_4's. Thus H' is isomorphic to F_{24}.

Now assume $g = b$. Then vertex f misses c and e and must see d or else $W(H')$ is a C_4. But then edge ce is a wing in three different P_4's.

If $g = d$ then f cannot see b or else edge ce is contained in three different P_4's. Vertex f also cannot see a or else H' contains a C_5. Thus H' is isomorphic to F_{22}.

Now we have to deal with the case that fg is disjoint to H. Then exactly one of the vertices f and g must see exactly one of the vertices c and e. We may assume that vertex g sees either c or e.

If ge is an edge then f must miss d or else ce is a wing in three different P_4's. Vertex g also must miss d as otherwise the four edges ab, cd, fg, ec induce a C_4 in $W(H')$. Now g must miss b or else edge cd is a wing in three different P_4's. This implies that neither ga nor fb nor fa can be an edge or else H' contains a C_5 or C_6 as an induced subgraph. This implies that H' is isomorphic to the graph F_{29}.

Now let us assume that gc is an edge. Then vertex d must see at least one of f and g or else the edges ab, cd, fg, ce induce a C_4 in $W(H')$. Vertex f cannot see d as otherwise edge ec belongs to three different P_4's. Therefore gd must be an edge. Now g must also see a and b or else ab is a wing in three different P_4's. But then ce is a wing in three different P_4's. $\diamond$

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

claim 2 *If $H = F_3$ then H' is isomorphic to F_{23}, F_{28} or F_{31}.*

Let the vertices of H be labeled a, b, c, d, e such that $abcd$ induces a P_4 and the vertex e is connected to c and d. We now have to add a wing fg to H that induces a P_4 together with edge ce to obtain the graph H'. Let us first assume that edge fg has vertex g with the graph H in common.

If $a = g$ then vertex f must see exactly one of c and e. If f sees e then it must also see b as otherwise H' contains a C_5. Now f cannot see d as otherwise $W(H')$ is a C_4. Therefore H' is isomorphic to F_{31}. If f sees c then it must also see d as otherwise $W(H')$ contains a C_4. But then f must also see b or else edge ab is a wing in three different P_4's. Thus H' is isomorphic to F_{23}.

Now assume $g = b$. Then vertex f misses c and e and must see d or else $W(H')$ is a C_4. Then f must a or else edge ab is a wing in three different P_4's. Now H' is isomorphic to F_{31}.

Now we have to deal with the case that fg is disjoint to H. Then exactly one of the vertices f and g must see exactly one of the vertices c and e. We may assume that vertex g sees either c or e.

If ge is an edge then f must miss b and g must miss a or else ce is a wing in three different P_4's. Now f must miss a as otherwise H' contains a C_6 or F_7 as an induced subgraph. Vertex g cannot see b or else edge ab is a wing in three different P_4's. If vertex f sees d then g must also see d or else the four edges ab, cd, fg, ec induce a C_4 in $W(H')$. But then bc is a wing in three different P_4's. If f misses d and g sees d then the four edges ab, cd, fg, ec induce a C_4 in $W(H')$. Thus H' must be isomorphic to F_{28}.

Now let us assume that gc is an edge. Then vertex g must miss a and vertex f must miss b or else ec is a wing in three different P_4's. Now g must see b and f must see a or else ab is a wing in three different P_4's. Vertex d can neither see g nor f or else ab is a wing in three different P_4's. But then the edges ab, cd, fg, ce induce a C_4 in $W(H')$. $\diamond$

claim 3 *If $H = F_4$ then H' is isomorphic to F_{26} or F_{31}.*

Let the vertices of H be labeled a, b, c, d, e such that $abcd$ induces a P_4 and the vertex e is connected to b and d. We now have to add a wing fg to H that induces a P_4 together with edge de to obtain the graph H'. Let us first assume that edge fg has vertex g with the graph H in common.

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

If $a = g$ then vertex f must see exactly one of d and e. If f sees e then it cannot see c as otherwise H' induces F_{16} or F_{21}. It also cannot see b or else edge cd is a wing in three different P_4's. But now H' is isomorphic to F_7. If f sees d then it must also see b or else H' contains an induced C_5. Now f must see c or else H' is isomorphic to F_{18}. But then H' is isomorphic to F_{26}.

Now assume $g = b$. Then vertex f misses d and e and must see c or else $W(H')$ is a C_4. But then edge de is a wing in three different P_4's.

Next assume $g = c$. Then vertex f misses d and e and must also miss b or else edge de is a wing in three different P_4's. But then fa must be an edge or else edge ab is a wing in three different P_4's. Now H' is isomorphic to F_{31}.

Now we have to deal with the case that fg is disjoint to H. Then exactly one of the vertices f and g must see exactly one of the vertices d and e. We may assume that vertex g sees either d or e.

If ge is an edge then g must miss a and f must miss b or else de is a wing in three different P_4's. If f sees a then g must see b or else H' contains an induced C_5. Now fc must be an edge or else fa belongs to three different P_4's. But then de belongs to three different P_4's. Thus fa cannot be an edge. But then ab is contained in three different P_4's independent of the existence of the edge gb.

Now let us assume that gd is an edge. Then f must miss b and c and g must miss a as otherwise edge ed is a wing in three different P_4's. Then gc must be an edge or else the egdes fg, cd, ab, ed induce a C_4 in $W(H')$. Now ab must not be a wing in three different P_4's and therefore g must see b. But then again ab is a wing in three different P_4's, a contradiction. ◇

claim 4 *If $H = F_5$ then H' is isomorphic to F_{23}, F_{24} or F_{27}.*

Let the vertices of H be labeled a, b, c, d, e such that $abcd$ induces a P_4 and the vertex e is connected to b, c and d. We now have to add a wing fg to H that induces a P_4 together with edge de to obtain the graph H'. Let us first assume that edge fg has vertex g with the graph H in common.

If $a = g$ then vertex f must see exactly one of d and e. If f sees e then it cannot see c as otherwise H' induces F_{16} or F_{17}. It must see b or else edge af is a wing in three different P_4's. Thus H' is isomorphic to F_{23}. If f sees d then it must also see b

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

or else H' contains an induced C_5. Now f must see c or else H' is isomorphic to F_{19}. But then H' is isomorphic to F_{27}.

Now assume $g = b$. Then vertex f misses d and e and must see c or else $W(H')$ is a C_4. Now H' is isomorphic to F_{23} or F_{24}.

Now we have to deal with the case that fg is disjoint to H. Then exactly one of the vertices f and g must see exactly one of the vertices d and e. We may assume that vertex g sees either d or e.

If ge is an edge then g must miss a and f must miss b or else de is a wing in three different P_4's. Now g must see b or else ab is a wing in three different P_4's. For the same reason fa must be an edge. Now fc must be an edge or else fa is a wing in three different P_4's. But then again fa is a wing in three different P_4's.

Now let us assume that gd is an edge. Then f must miss b and g must miss a as otherwise edge ed is a wing in three different P_4's. As $W(H')$ must not contain a C_4 as induced subgraph, at least one of f, g must see c. If f sees c then f must also see a or else edge ab is a wing in three different P_4's. But then fa is a wing in three different P_4's. Thus f cannot see c but g must see c. But now fg is a wing in three different P_4's, independent of the existence of the edge gb. $\diamond$

claim 5 *If $H = F_6$ then H' is isomorphic to $\overline{C}_6$, F_{25}, F_{26}, F_{27}, F_{30} or F_{31}.*

Let the vertices of H be labeled a, b, c, d, e such that $abcd$ induces a P_4 and the vertex e is connected to a, b and d. We now have to add a wing fg to H that induces a P_4 together with edge ae to obtain the graph H'. Let us first assume that edge fg has vertex g with the graph H in common.

If $c = g$ then vertex f must see exactly one of a and e. If f sees e then H' is isomorphic to F_{18}, F_{19}, F_{26} or F_{27}. If f sees a then it must also see d or else H' contains an induced C_5. But then H' is isomorphic to $\overline{C}_6$ or F_{25}.

Now assume $g = d$. Then vertex f must see at least one of b, c as otherwise edge fd is a wing in three different P_4's. But then H' is isomorphic to F_{18}, F_{19} or F_{31}.

Now we have to deal with the case that fg is disjoint to H. Then exactly one of the vertices f and g must see exactly one of the vertices a and e. We may assume that vertex g sees either a or e.

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

If ge is an edge then g must miss c and f must miss d or else ae is a wing in three different P_4's. Now g must see d or else cd is a wing in three different P_4's. For the same reason fc must be an edge. Now fb must be an edge or else edge cf is a wing in three different P_4's. Then gb must be an edge or else the edges fg, ab, cd, ae induce a C_4 in $W(H')$. Now H' is isomorphic to F_{30}. Now let us assume that ga is an edge. Then f must miss d and g must miss c as otherwise edge ea is a wing in three different P_4's. Similarly, g must miss d and b or else edge cd is a wing in three different P_4's. Now fb must be an edge or else $W(H')$ contains a C_4. Vertex f cannot see c as otherwise H' contains an induced C_6. But then cd is a wing in three different P_4's. ◇

claim 6 *If $H = F_8$ then H' is isomorphic to F_{22}.*

The wing-graph of F_8 is a P_5. Therefore we have $H = H'$. ◇

claim 7 *If $H = F_9$ then H' is isomorphic to F_{30}.*

Let the vertices of H be labeled a, b, c, d, e, f such that $abcd$ induces a P_4, the vertex e is connected to d and vertex f is connected to b, c and e. We now have to add a wing gh to H that induces a P_4 together with edge ef to obtain the graph H'. Let us first assume that edge gh has vertex g with the graph H in common.

If $a = g$ then vertex h must see exactly one of e and f. If h sees e then it must also see b or else H' contains an induced C_5. Now dh must be an edge or else edge de is contained in three different P_4's. Finally h must also see c as otherwise the four edges ah, dc, ba, ef induce a C_4 in $W(H')$. But then H' is isomorphic to F_{30}. Next assume that h sees f. Then it must also see d or else edge de is a wing in three different P_4's. But then again edge de is a wing in three different P_4's.

Now assume $g = b$. Then vertex h misses e and f and it also must miss d and c or else edge ef is a wing in three different P_4's. But then $W(H')$ contains a C_4.

If $g = c$ then h misses e and f and must also miss d or else edge ef is a wing in three different P_4's. But then edge de is a wing in three different P_4's.

Finally assume $g = d$. Then h misses e and f and must also miss b and c or else edge ef is a wing in three different P_4's. But then edge hd is a wing in three different P_4's.

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

Now we have to deal with the case that gh is disjoint to H. Then exactly one of the vertices g and h must see exactly one of the vertices e and f. We may assume that vertex g sees either e or f.

If ge is an edge then h must miss b, c, d and g must miss a or else ef is a wing in three different P_4's. If gc is an edge then g must see d and b or else edge hg is a wing in three different P_4's. But then again hg is a wing in three different P_4's. If gb is an edge then hg is a wing in three different P_4's. Finally gd cannot be an edge since otherwise H' contains F_{31} as a proper induced subgraph. Thus g sees neither b nor c nor d. But then edge ge is a wing in three different P_4's. Now let us assume that gf is an edge. Then h must miss d as otherwise edge fe is a wing in three different P_4's. Vertex g must see d or else ed is a wing in three different P_4's. But then again edge de is a wing in three different P_4's. ◇

claim 8 *If $H = P_6$ then H' is isomorphic to C_7, P_8, F_{28} or F_{29}.*

Let $H = abcdef$ be a P_6. We now have to add a wing gh to H that induces a P_4 together with edge ef to obtain the graph H'. Let us first assume that edge gh has vertex g with the graph H in common.

If $a = g$ then vertex h must see exactly one of e and f. If h sees f then it must miss b and c or else edge ef is a wing in three different P_4's. Vertex f must also miss d as otherwise H' contains an induced C_5. Thus H' is isomorphic to C_7. Now assume that h sees e. Then it must miss b and c or else edge ef is a wing in three different P_4's. But then H' contains C_5 or C_6 as an induced subgraph.

If $g = b$ then h must see exactly one of e and f. In both cases it must miss c or else ef is a wing in three different P_4's. If h sees f then H' contains either C_6 or F_7 as an induced subgraph. If h sees e then it must also see d or else H' contains C_5 as an induced subgraph. Now ha must be an edge or else edge ab is a wing in three different P_4's. But then cd is a wing in three different P_4's.

Now assume $g = c$. Then h must see exactly one of e and f. In both cases h cannot see a or b as otherwise edge ef is a wing in three different P_4's. Assume hf is an edge. Then h must see d or else H' contains an induced C_5. But then $W(H')$ contains a C_4. Now assume he is an edge. Then $W(H')$ contains a C_4.

Finally assume $g = d$. If h misses c then it also must miss a or else edge cd is a wing in three different P_4's. Vertex h must also miss b as otherwise $W(H')$ contains

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

a C_4. But then H' is isomorphic to F_{29}. If h sees c then it cannot see both a and b as otherwise edge de is a wing in three different P_4's. Vertex h also cannot see exactly one of a or b as otherwise $W(H')$ contains a C_4. Thus h misses both a and b and H' is isomorphic to F_{28}.

Now we have to deal with the case that gh is disjoint to H. Then exactly one of the vertices g and h must see exactly one of the vertices e and f. We may assume that vertex g sees either e or f.

If ge is an edge then g must miss a, b, c and h must miss d or else ef is a wing in three different P_4's. Now gd must be an edge or else edge cd is a wing in three different P_4's. For the same reason hc must be an edge. Then hb must be an edge or else bc is a wing in three different P_4's. But then H' contains F_{31} as an induced subgraph.

Now assume gf is an edge. Then g must miss a, b, c and h must miss d or else ef is a wing in three different P_4's. Vertex g must miss d or else edge cd is a wing in three different P_4's. Then h must miss a, b, c or else H' contains C_6, C_7 or C_8 as an induced subgraph. Thus H' is isomorphic to P_8.

◇

□

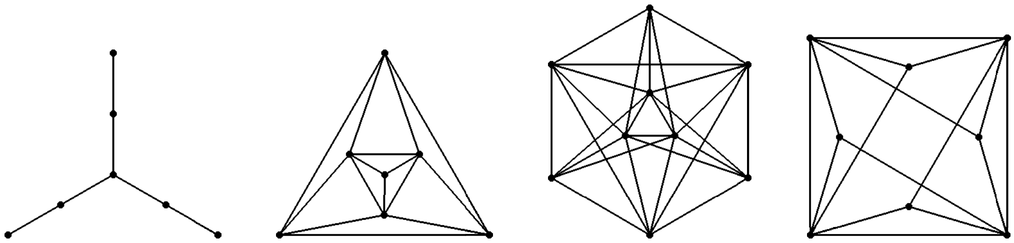
5 Proof of the main theorem

Theorem 1 *Let G be a P_4-dense graph whose wing-graph is a cycle of length at least seven. Then G is an odd cycle of length at least five or the complement of a cycle of length at least five or one of the graphs $F_7, F_{31}, F_{32}, F_{33}, F_{34}, F_{35}$ (see Figure 4).*

Proof. Let G be a P_4-dense graph whose wing-graph is a cycle of length at least five. Let the wings in the wing-graph of G be labeled $1, 2, 3, \ldots$ consecutively such that consecutive wings are consecutive vertices in the cycle $W(G)$. We now consider the subgraph G_i of G that is induced by the wings $i, i+1, i+2, i+3$.

From Lemma 3 we know that each G_i must be isomorphic to one of the graphs C_5, $\overline{C}_6$, C_7, F_7, P_8, F_{22}-F_{31}. If G_i is isomorphic to C_5, $\overline{C}_6$, C_7, F_7 or F_{31} then we are immediately done as these graphs have a cycle of length at least five as their wing-graph. Thus we may assume that G_i is isomorphic to P_8 or F_{22}-F_{30}. Clearly, G_{i+1} also

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66



F_{32} F_{33} F_{34} F_{35}

Figure 4: Graphs whose wing-graph is a long cycle.

must be one of these graphs and has three wings in common with G_i. We will now study which combinations of graphs are possible for G_i and G_{i+1}. Some of the graphs in P_8, F_{22}-F_{30} have several choices for four consecutive wings that induce the graph. In these cases we may choose any of the possible four wings as we know that for some j the graph G_j is exactly the graph induced by these four wings (after possibly reversing the numbering of the wings). We have indicated which four wings we will assume to be the wings $i, i+1, i+2, i+3$ in Figure 5. We name these graphs with the chosen wings $W_1, \ldots, W_{10}$ and use the labeling of the vertices as shown in Figure 5. All edges drawn in bold are wings in the graphs.

claim 1 *If G_i is isomorphic to W_1 then G is an odd cycle of length at least 9.*

If G_i is the graph W_1 then the graph induced by the wings $i+1, i+2, i+3$ is a P_6. Therefore G_{i+1} must be one of the graphs F_{28}, F_{29} and P_8. Assume G_{i+1} is isomorphic to F_{28}. Then there must exist a vertex x that sees e and f and none of c, d, g, h. But then edge cd is a wing in three different P_4's. Now assume G_{i+1} is isomorphic to F_{29}. Then there must exist a vertex x that sees f and none of c, d, e, g, h. But then de is a wing in three different P_4's. Thus G_{i+1} must be isomorphic to P_8. This implies that any four consecutive wings in G induce a P_8 and therefore G must be an odd cycle of length at least 9. $\diamond$

claim 2 *G_i cannot be isomorphic to W_2.*

If G_i is isomorphic to W_2 then the graph G_{i+1} must be isomorphic to F_{23} or F_{28} as no other graph contains the graph that is induced by the wings $i+1, i+2, i+3$ in W_2. If G_{i+1} is isomorphic to F_{23} then there must exist a vertex x that sees c, d, e, f and misses g. But then edge cg is a wing in three different P_4's. If G_{i+1} is isomorphic

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

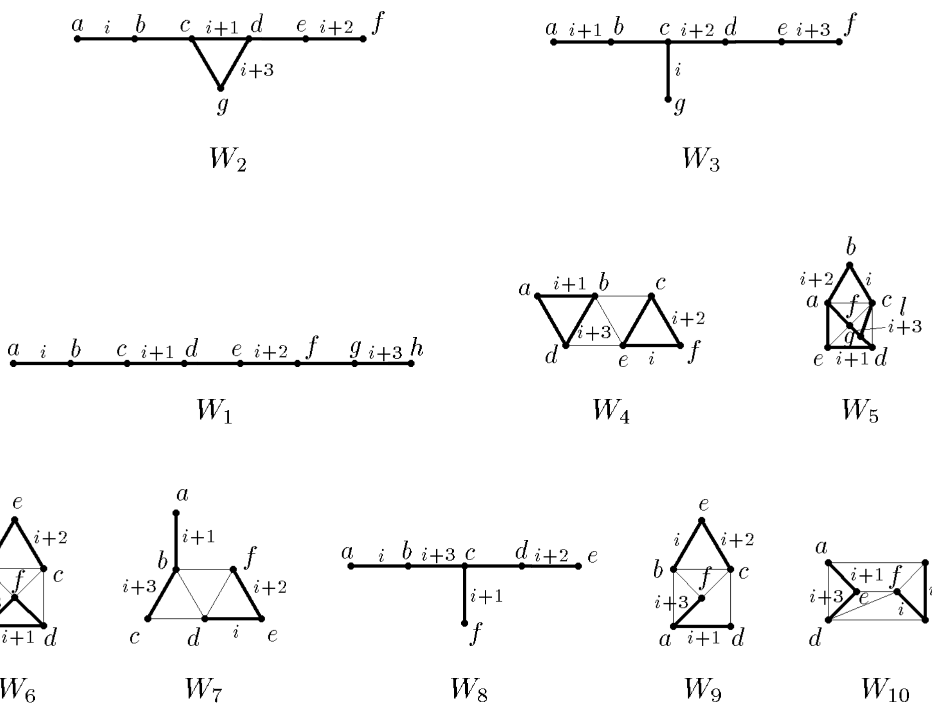


Figure 5: Graphs with a P_4 in their wing-graph.

to F_{28} then there must exist two adjacent vertices x and y such that x sees g and both x, y miss c, d, e, f. Since G must not contain F_1 as an induced subgraph we must have edge xb. But then edge dg is a wing in three different P_4's. ◇

claim 3 *G_i cannot be isomorphic to W_3.*

If G_i is the graph W_3 then the graph induced by the wings $i+1, i+2, i+3$ is a P_6. Therefore G_{i+1} must be isomorphic to F_{29} as graphs F_{28} and P_8 are already ruled out by claims 1 and 2. If G_{i+1} is isomorphic to F_{29} then there must exist a vertex x that sees d and misses a, b, c, e, f. As xd must not be a wing in three different P_4's we must have edge xg. But then edge de is a wing in three different P_4's. ◇

claim 4 *G_i cannot be isomorphic to W_4.*

If G_i is isomorphic to W_4 then the graph G_{i+1} must be isomorphic to F_{23} or F_{28} as no other graph contains the graph that is induced by the wings $i+1, i+2, i+3$ in W_4.

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

The graph F_{28} is already ruled out by claim 2 thus G_{i+1} must be isomorphic to F_{23}. Then there must exist a vertex x that sees a, b, c, f and misses d. But then edge ad is a wing in three different P_4's. $\diamond$

claim 5 *G_i cannot be isomorphic to W_5.*

If G_i is isomorphic to W_5 then the only graph containing the graph induced by the wings $i+1, i+2, i+3$ is F_{30}. Thus G_{i+1} must be isomorphic to F_{30}, i.e. there must exist a vertex x that sees a, b, d, e, g and misses f. If xc is not an edge then edge fc is a wing in three different P_4's. If xc is an edge then the edges ab, gf, xb, ef, cb, de induce a C_6 in $W(G)$. $\diamond$

claim 6 *If G_i is isomorphic to W_6 then G is isomorphic to F_{33}.*

If G_i is isomorphic to W_6 then the only graphs containing the graph induced by the wings $i+1, i+2, i+3$ that are not ruled out by the preceding claims are F_{24} and F_{27}. Assume first that whenever four consecutive wings in G induce the graph F_{27} then the next four consecutive wings do not induce F_{27}. Thus G_{i+1} must be isomorphic to F_{24}. Then there must exist a vertex x that sees c and d and misses a, e, f. Vertex x must also miss b or else edge be is a wing in three different P_4's. Now G_{i+4} (induced by the wings cx, ba, dx, bf) is isomorphic to F_{27} and therefore by our assumption G_{i+5} must be isomorphic to F_{24}. Thus there exists a vertex y that sees a, d and misses b, f, x. Then y must see e or else edge eb is a wing in three different P_4's. But then vertices e, b, f, d, y induce a C_5 in G. Now assume that G_{i+1} is isomorphic to F_{27}. Then there exists a vertex x that sees a, c, d, e and misses f.Then x must see b or else G contains a C_5 as induced subgraph.But then G is isomorphic to F_{33}. $\diamond$

claim 7 *G_i cannot be isomorphic to W_7.*

If G_i is isomorphic to W_7 then the only graphs containing the graph induced by the wings $i+1, i+2, i+3$ that are not ruled out by the preceding claims are F_{22} and F_{24}. Assume first that G_{i+1} is isomorphic to F_{22}. Then there must exist a vertex x that sees a and misses c, b, e, f. Then xd must be an edge or else xa is a wing in three different P_4's. But then again xa is a wing in three different P_4's. Now assume that G_{i+1} is isomorphic to F_{24}. Then there must exist a vertex x that sees a, b, f, e and misses c. Then x must see d or else G_{i+3} (induced by the wings xf, cd, xe, cb) is isomorphic to W_6, contradicting the previous claim. Now G_{i+2} also must be isomorphic to F_{24}. This

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

implies the existence of a vertex y that sees b, f and misses c, e, x. Then y must see d or else the six edges ed, by, xe, cb, fe, ab induce a C_6 in $W(G)$. Then y must also see a or else ax is a wing in three different P_4's. But now xe is a wing in three different P_4's. ◇

claim 8 *If G_i is isomorphic to W_8 then G is isomorphic to F_{32}.*

If G_i is isomorphic to W_8 then the only graph containing the graph induced by the wings $i+1, i+2, i+3$ that is not ruled out by the preceding claims is F_{22}. Thus there must exist a vertex x that sees f and misses b, c, d, e. Then x must also miss a or else G contains a C_5 as an induced subgraph. Now G is isomorphic to F_{32}. ◇

claim 9 *If G_i is isomorphic to W_9 then G is isomorphic to F_{34} or F_{35}.*

If G_i is isomorphic to W_9 then the only graph containing the graph induced by the wings $i+1, i+2, i+3$ that is not ruled out by the preceding claims is F_{26}. Thus there must exist a vertex x that sees a, e, c, d and misses f.

Suppose first that x misses b. Then the graph induced by the edges fa, xe, bf is isomorphic to F_6 and therefore G_{i+3} must be isomorphic to F_{25} or F_{26}. If G_{i+3} is isomorphic to F_{25} then there must exist a vertex y such that y sees e, x, f, a and misses b. Then y must miss d or else edge be is a wing in three different P_4's. But then edge da is a wing in three different P_4's. Now assume that G_{i+3} is isomorphic to F_{26}. Then there must exist a vertex y such that y sees a, b, x and misses e. Then y cannot see c or else edge ec is a wing in three different P_4's. Vertex y also cannot see d or else edge dy is a wing in three different P_4's. Thus G is isomorphic to F_{35}.

Now suppose that xb is an edge. Then the graph induced by ex, af, xe is isomorphic to F_6 and therefore G_{i+2} must be isomorphic to F_{25} or F_{26}. Assume first that G_{i+2} is isomorphic to F_{26}. Then there must exist a vertex y that sees c, x, f and misses e, a. Now yd must be an edge or else ad is a wing in three different P_4's. Vertex y must also see b as otherwise the edges xe, yf, eb, ad, ce, fa induce a C_6 in $W(G)$. Now the graph induced by the wings fa, xe, fy implies that G_{i+3} is isomorphic to F_{26}. Thus there must exist a vertex z that sees e, x, a, f and misses y. Now zc must be an edge or else edge ec is a wing in three different P_4's. Similarly zd must be an edge as otherwise edge ad is a wing in three different P_4's. Now zb must be an edge or else dz is a wing in three different P_4's. But then the edges $fy, ez, dy, be, ad, ec, fa, xe$ induce a C_8 in

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

$W(G)$. Now assume that G_{i+2} is isomorphic to F_{25}. Then there must exist a vertex y that sees a, f, c, e and misses x. Now yd must be an edge or else ad is a wing in three different P_4's. Then y must see b or else edge fy is a wing in three different P_4's. Now the edges fy, xd, fb induce F_6. Therefore the graph G_{i+5} must be isomorphic to F_{25} or F_{26}. Assume first that G_{i+5} is isomorphic to F_{26}. Then there must exist a vertex z that sees b, d, y and misses x, f. Now za must be an edge or else fa is a wing in three different P_4's. Similarly edge ze must exist as otherwise ex is a wing in three different P_4's. Now zc must be an edge or else the vertices a, f, c, e, z induce a C_5. But now edge fa is a wing in three P_4's. Now assume that G_{i+5} is isomorphic to F_{25}. Then there must exist a vertex z that sees y, d, f, x and misses b. Now ze must be an edge or else xe is a wing in three different P_4's. Now za must be an edge or else the vertices a, b, e, z, d induce a C_5. Finally zc must be an edge or else edge dz is a wing in three different P_4's. Now G is isomorphic to F_{34}. ◇

By the preceding claims we finally have to deal with the case that for all i the graph G_i is isomorphic to F_{25}.

claim 10 *If for all i the graph G_i is isomorphic to W_{10} then G is the complement of a cycle of length at least 7.*

Let G_i be isomorphic to W_{10} and G_{i+1} be isomorphic to F_{25}. Then there must exist a vertex x that sees a, b, c, e and misses d. If x misses f then G is isomorphic to $\overline{C}_7$. If x sees f then there must exist a vertex y that sees b, c, d, e and misses x. Then y must see a or else edge ea is a wing in three different P_4's. If y misses f then G is isomorphic to $\overline{C}_8$. Otherwise we get by induction that G is isomorphic to the complement of a cycle of length at least 9. ◇

□

As corollaries of the main theorem we get a characterization of all graphs whose wing-graph is a C_5 or C_6. We also obtain that the conjecture of Hoàng is true with the only exception of the graph F_{34}.

Corollary 3 *C_5 is the unique P_4-dense graph whose wing-graph is a C_5.*

□

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

Corollary 4 *Let G be a P_4-dense graph whose wing-graph is a C_6. Then G is a $\overline{C}_6$, F_7 or one of the graphs $F_{31} - F_{33}$ (see Figure 4).*

□

Corollary 5 (Hoàng's Conjecture) *Let G be a P_4-dense graph whose wing-graph is an odd cycle of length at least five. Then G is an odd cycle of length at least five or the complement of an odd cycle of length at least five or the graph F_{34}.*

□

As another consequence of the main theorem we get similarly to Hoàng's conjecture a characterization of all graphs whose wing-graph is an *even* cycle.

Corollary 6 *Let G be a P_4-dense graph whose wing-graph is an even cycle of length at least 8. Then G is the complement of an even cycle of length at least five or the graph F_{35}.*

□

6 Acknowledgement

I am grateful to an anonymous referee for pointing out an error in an earlier version of the proof of Theorem 1.

References

[1] V.Chvátal, *A Semi–Strong Perfect Graph Conjecture,* Ann. Discrete Math. 21 (1984), 279–280

[2] V.Chvátal, *On the P_4-structure of perfect graphs III. Partner Decompositons,* J. Combin. Theory Ser. B 43 (1987), 349–353

[3] V.Chvátal, C.T.Hoàng, *On the P_4–structure of perfect graphs I. Even decompositions,* J. Combin. Theory Ser. B 39 (1985), 209–219

This paper appeared in: Journal of Graph Theory 31 (1999), 51-66

[4] V.Chvátal, W.J.Lenhart, N.Sbihi, *Two–colourings that decompose perfect graphs,* J. Combin. Theory Ser. B 49 (1990), 1–9

[5] V.Chvátal, *Problems concerning perfect graphs,* Manuscript, Princeton conference on perfect graphs, July 1993

[6] M.C.Golumbic, *Algorithmic Graph Theory and Perfect Graphs,* Academic Press, New York, 1980

[7] R.B.Hayward, W.J.Lenhart, *On the P_4–structure of perfect graphs. IV. Partner graphs,* J. Combin. Theory Ser. B 48 (1990), 135–139

[8] C.T.Hoàng, *On the P_4–structure of perfect graphs II. Odd decompositions,* J. Combin. Theory Ser. B 39 (1985), 220–232

[9] C.T.Hoàng, *On the two–edge–colourings of perfect graphs,* J. Graph Theory 19 (1995), 271–279

[10] C.T.Hoàng, S.Hougardy, F.Maffray, *On the P_4-structure of perfect graphs V. Overlap graphs,* J. Combin. Theory Ser. B 67 (1993), 212–237

[11] S.Hougardy, V.B.Le, A. Wagler, *Wing-triangulated graphs are perfect,* J. Graph Theory 24 (1997), 25–31

[12] B.A.Reed, *A Semi–Strong Perfect Graph Theorem,* J. Combin. Theory Ser. B 43 (1987), 223–240