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A Conjecture of Kauffman on Amphicheiral Alternating Knots

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Abstract

We give a counterexample to the following conjecture of Kauffman [2]:

Conjecture *Let K be an amphicheiral alternating knot. Then there exists a reduced alternating knot-diagram D of K, such that $G(D)$ is isomorphic to $G^*(D)$, where $G(D)$ is a checkerboard-graph of D and $G^*(D)$ its dual.*

Keywords: knots, links, alternating knots, amphicheirality, checkerboard graph

1 Introduction

Lou Kauffman conjectured [2] (revised in [3]) that every amphicheiral alternating knot can be drawn so that the checkerboard-graph of the knot diagram is self-dual. This does not hold. We will give a counterexample.

Section 2 describes some conclusions of the "Flyping Conjecture", proved recently by Menasco and Thistlethwaite [4], that characterizes alternating projections of a link.
Equipped with these tools we show that the knot on 14 crossings given in Section 3 is a counterexample to the conjecture, stated above.

Furthermore we prove

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Theorem 1 *Let K_1 be an amphicheiral alternating knot with reduced alternating diagram D_1 and checkerboard graph G_1. Then there exists an alternating knot K_2 with diagram D_2 and checkerboard-graph G_2 such that G_2 is as a graph isomorphic to G_1 and isomorphic to its dual G_2^*.*

We thank Lou Kauffman for pointing us to his conjecture.

2 Alternating knot projections

Our terminology is standard and we assume basic knowledge of knot theory (see [5]).
We regard knot diagrams as projections on S^2 rather than on $\mathbb{R}^2$. So two knot diagrams are equivalent if there is an autohomeomorphism of S^2 which maps one to the other.
Alternating knot diagrams are classified by the famous "Tait Flyping Conjecture", which has been proved by Menasco and Thistlethwaite [4]:

Proposition 2.1 *Let $D_1 := D(K_1)$ and $D_2 := D(K_2)$ be two alternating reduced (i.e. without nugatory crossing, see Figure 1) diagrams of prime knots. Then K_1 and K_2 are equivalent (ambient isotopic) if and only if D_1 may be transformed into D_2 by flypes (see Figure 2).*

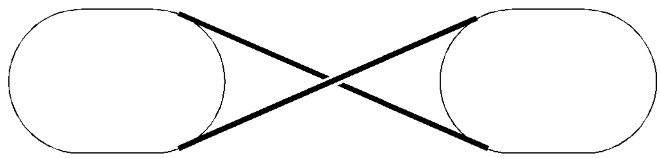

Figure 1: Nugatory crossing

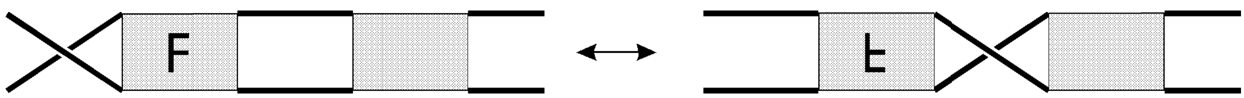

Figure 2: Flypes

To use the "Flyping Conjecture" in our context we construct - as it is standard - to every knot diagram two signed plane (embedded in S^2) *checkerboard-graphs*:

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The faces of a knot diagram can be colored black and white so that adjacent faces have different colors. Now the vertices of the graphs correspond to the black (resp. white) faces of the knot diagram. Two vertices are joined if the corresponding faces share a crossing. We obtain a sign according to the rule in Figure 3.
It can be shown (see for example [5]) that the knot is uniquely determined by one of its plane checkerboard-graphs.

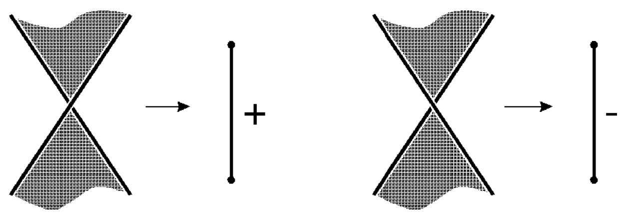

Figure 3: Signs of the checkerboard-graph coming from black faces

Of course the two checkerboard-graphs (as unsigned graphs) are dual and dual edges have different signs.
It is easy to see that in an alternating knot diagram all edges of a checkerboard-graph bear the same sign. Therefore we can associate with every alternating knot diagram a unique checkerboard-graph if we take the graph having only positive signs. We will call this a *positive checkerboard-graph.*
Now we have

Corollary 2.2 *Two alternating reduced diagrams of prime knots are projections of equivalent knots if and only if their positive checkerboard-graphs are transformable into each other by flypes of type F_1 and F_2 given by Figure 4.*

Remark It is necessary to regard flypes as operations on plane graphs, because knots are uniquely determined by their embeddings and not by their isomorphism class of a checkerboard-graph.

3 A counterexample to a conjecture of Kauffman

In [2] (revised in [3]) Lou Kauffman conjectured the following:

Conjecture *Let K be an amphicheiral alternating knot. Then there exists a reduced alternating knot-diagram D of K, such that $G(D)$ is isomorphic to $G^*(D)$, where $G(D)$ is a checkerboard-graph of D and $G^*(D)$ its dual.*

First of all we have:

Theorem 1 *Let K_1 be an amphicheiral alternating knot with reduced alternating diagram D_1 and checkerboard graph G_1. Then there exists an alternating knot K_2 with diagram D_2 and checkerboard-graph G_2 such that G_2 is as a graph isomorphic to G_1 and isomorphic to its dual G_2^*.*

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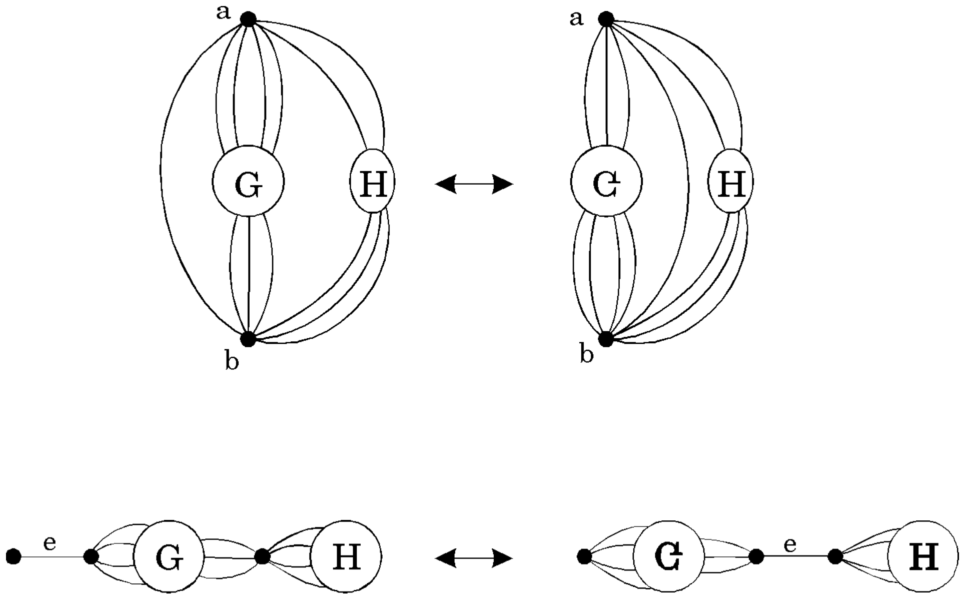

Figure 4: Flypes of type F_1 and F_2

Proof Amphicheirality means that G_1 may be transformed into its dual G_1^* by flypes and vice versa.
If a plane graph H_1 is transformable into a graph H_2 by flypes, then H_2 may be embedded so that its dual is isomorphic to the dual of H_1. To see this, note that if a graph H_1 is transformed to a graph H_1' by a flype of type F_1 (see Figure 4) then an embedding of H_1' whose dual is isomorphic to the dual of H_1 is obtained by simply embedding the edge $\overline{ab}$ on the same position as in H_1.
Similarly if H_1 is transformed to H_1' by a flype of type F_2, then by flipping over the part G of the graph H_1' we obtain an embedding of H_1' whose dual is isomorphic to the dual of the embedding of H_1.
Now G_1^* is transformable to G_1 by flypes and therefore G_1 may be embedded so that its dual is isomorphic to $(G_1^*)^* = G_1$.
Since we can construct to every plane graph a link that has this graph as its checkerboard-graph, it remains to show that the new link is indeed a knot. This means that two links with isomorphic checkerboard graphs have the same number of components.
This follows from the well-known close relation between the Jones polynomial V_L of an alternating link L and the Tutte polynomial χ_G of a checkerboard-graph G of L, which is an invariant of the abstract graph. (For a good account of the Jones polynomial see [6].)
The number of components of a link can be deduced from the evaluation of the Jones polynomial at 1. Precisely we have $V_L(1) = (-2)^{c(L)-1}$, where $c(L)$ is the number of components of L and $V_L(1) = \pm\chi_G(-1,-1)$. □

Unfortunatly the theorem does not proof the conjecture but it gives us to every

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counterexample a construction and embedding of an alternating knot which has the property conjectured for alternating amphicheiral knots.
A counterexample is given by the following:

Theorem 2 *There exists an amphicheiral alternating knot admitting no alternating embedding with a self-dual checkerboard-graph.*

Proof In Figure 5 and Figure 6 a counterexample K and the positive checkerboard-graphs G_1 and G_1^* of the knot-diagram and its mirror-image are given. Remember that the two positive checkerboard-graphs must be dual.
By a result of Menasco [8] an alternating diagram is a projection of a prime knot if its checkerboard-graph is 2-connected. Therefore our knot is prime.

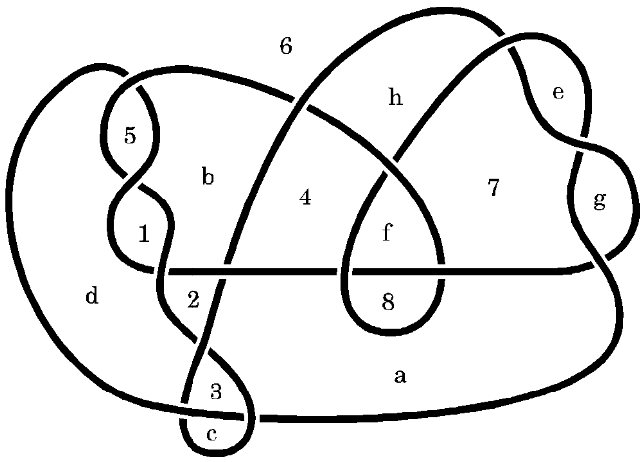

Figure 5: Counterexample

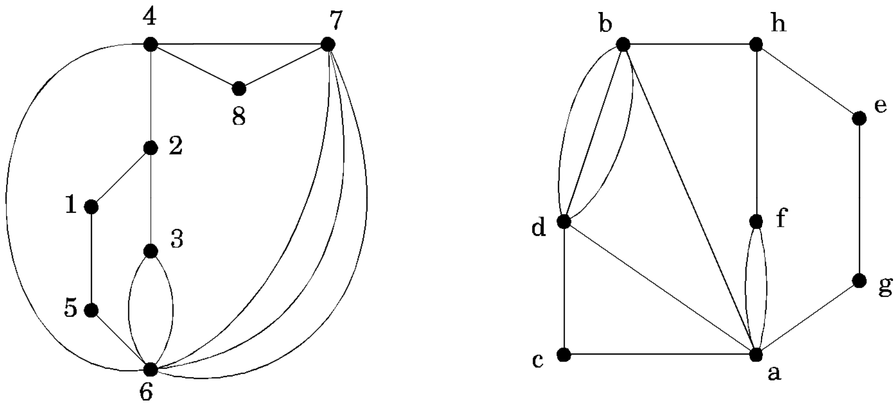

Figure 6: Checkerboard-graphs G_1 and G_1^*

Figure 7 shows checkerboard-graphs G_2, G_3 and G_4 of different embeddings of K. Four additional embeddings G_1' G_2', G_3' and G_4' are obtained by transforming the

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graphs by flypes involving the vertices 2, 3 and 6 in G_1 and G_4 and 2, 3 and 4 in G_2 and G_3.

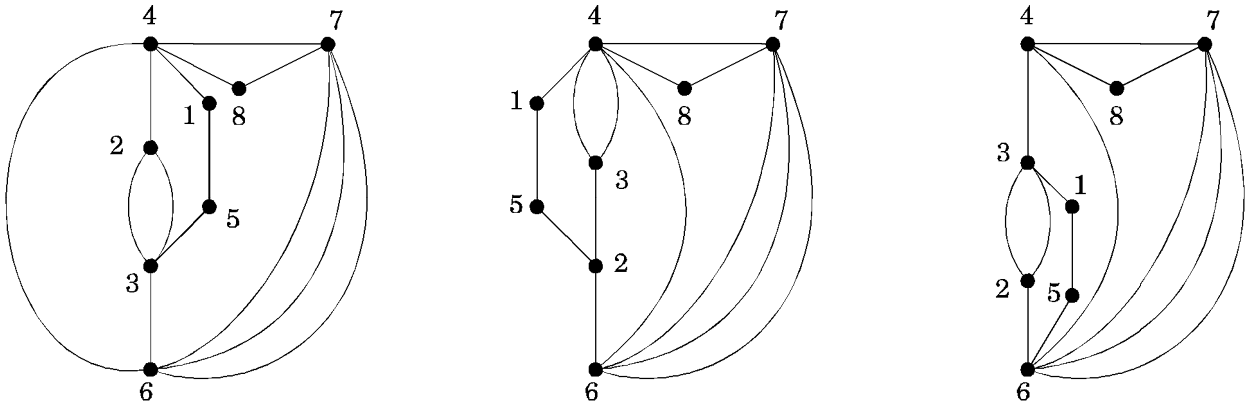

Figure 7: Checkerboard-graphs G_2, G_3 and G_4

According to the Flyping Conjecture these are - up to autohomeomorphism - the only possible different embeddings on the sphere.
Now it is easy to see that G_1^* and G_2' are determing the same knot. Therefore K is amphicheiral.
It remains to show that none of the eight graphs is isomorphic to its dual. Remembering the transformation from G_x to G_x' does not change the isomorphism-class of their duals it is easy to find vertices of some valencies in every graph that do not occur in the same number in the duals. The valency of a vertex in the dual of a graph of course is equal to the number of vertices adjacent to the face that represents this vertex. □

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