The Complexity of Some Problems on Very Sparse Graphs

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Abstract. We study the complexity of the problems DOMINATING SET, MAX CUT, VERTEX FEEDBACK SET, STEINER TREE, HAMILTONIAN CIRCUIT, and CHROMATIC INDEX on graphs G of bounded maximum degree and large girth. All results are essentially best possible. We also construct regular class-one graphs of large girth and small order. Finally, we point out how vertex resp. edge feedback sets of size $\mathcal{O}(\log n)$ can be used to solve MAX CUT, INDEPENDENT SET, NODE COVER, DOMINATING SET, VERTEX FEEDBACK SET and STEINER TREE in polynomial time.

1 Introduction

Let $G = (V, E)$ be a graph with node set V and edge set E. The order of G, i.e. the number of its vertices, is denoted by $|G|$ or $n = n(G)$, the number of its edges by $m = m(G)$ or just by $|E|$. The degree of a node $v \in V$, i.e. the number of neighbors of v in G, is denoted by $d(v)$, the maximum degree of a node in G by $\Delta(G)$. A graph every vertex of which has degree r is called r-regular and cubic if $r = 3$. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G and can be computed in time $\mathcal{O}(mn)$. For an introduction to the theory of computational complexity and the notion of NP-completeness we refer the reader to [GJ79, Pap94].

Bounding the maximum degree in optimization problems on graphs does often not affect their hardness, cf. e.g. [Joh85]. For a constant $k \in \mathbb{N}$, a graph G of maximum degree k is sparse in the sense that G has at most $k|G|/2$ edges. The girth is another graph parameter controlling the sparseness of a graph – a graph G of girth $g \geq 2h + 2$, $h \in \mathbb{N}$, has at most $\left(\frac{n}{2}\right)^{1+1/h} + 2^h \cdot \left(\frac{n}{2}\right)^{1-1/h}$ edges [ES82]. As graphs of large girth look like trees locally one might think that a hard problem might be easier to solve on graphs of large girth. Nevertheless, as we show in this paper, several well-known NP-hard problems on graphs remain NP-hard on graphs with large girth although being solvable in linear time on trees.

More specifically, we prove the NP-completeness of DOMINATING SET, MAX CUT, VERTEX FEEDBACK SET and STEINER TREE on graphs of girth $g(G) \geq |G|^r$ for any fixed r, $0 \leq r < 1$ (Section 2), the NP-completeness of HAMILTONIAN CIRCUIT on graphs of girth $g(G) \geq |G|^r$ for any fixed r, $0 \leq r < 1/2$, giving a polynomial time algorithm for graphs G of girth $g(G) > 2|G|^{1/2}$ (Section 3), and the NP-completeness of CHROMATIC

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INDEX on r-regular graphs of girth $g(G) \geq c \cdot \log|G|/\log(r-1)$, for any fixed $0 \leq c < 1$ and integer $r \geq 3$ (Section 4). The last result is best possible up to the constant c.

In all cases one may additionally impose the restriction to graphs of maximum degree 3 (resp. 6 in case of VERTEX FEEDBACK SET), thus forcing the graphs to be sparse both in a global and in a local way.

In a companion paper [EHK96] we have shown that an analogous result holds for the problem GRAPH COLORABILITY.

In Section 5 we show how the problems MAX CUT, INDEPENDENT SET, NODE COVER resp. DOMINATING SET can be solved in polynomial time given a vertex resp. edge feedback set of size $\mathcal{O}(\log n)$ for the graph at hand. As such a feedback set can be constructed in polynomial time in graphs G with girth $g(G) \geq \frac{n \text{llog} n}{\log n}$ this yields polynomial time algorithms for these problems as well as for the problems VERTEX FEEDBACK SET and STEINER TREE in graphs with girth $g(G) \geq \frac{n \text{llog} n}{\log n}$.

2 DOMINATING SET, MAX CUT, VERTEX FEEDBACK SET and STEINER TREE

In the case of DOMINATING SET we are given a graph $G = (V, E)$, and the problem is to find a smallest node set $S \subseteq V$ such that every vertex from $V \backslash S$ has at least one neighbor in S. The smallest possible size of such a set S is called the domination number $\gamma(G)$. In the case of MAX CUT we are given a graph $G = (V, E)$ and the problem is to find a partition $V = X\dot{\cup}Y$ of the node set such that the number $|E(X,Y)| := |\{\{x,y\} \in E : x \in X, y \in Y\}|$ of crossing edges is maximized. The largest possible size of such a set of edges is denoted by $mc(G)$. A vertex set $F \subseteq V$ which intersects all cycles of $G = (V, E)$ (i.e. $G[V \setminus F]$ is a forest) is called a vertex feedback set. The problem VERTEX FEEDBACK SET is to find the minimal size $vfs(G)$ of a vertex feedback set in a given graph G. In case of STEINER TREE, finally, we are given a graph $G = (V, E)$ and a set $T \subseteq V$ of so-called terminal vertices. A set $F \subseteq E$ of edges such that F induces a subgraph of G which is a tree and covers all the vertices from T is called a Steiner tree for G and T. The problem is to find the minimum number $St(G,T)$ of edges of a Steiner tree F for G and T. In case of the WEIGHTED STEINER TREE problem we are additionally given edge weights $w : E \rightarrow \mathbb{Q}^+$ and try to minimize $\sum_{e \in F} w(e)$. PLANAR DOMINATING SET is the problem DOMINATING SET restricted to planar graphs; analogously for the other problems.

Theorem 2.1 *For a rational number r, $0 \leq r < 1$, and an integer Δ let $\mathcal{G}_{\Delta,r}$ ($\mathcal{BPG}_{\Delta,r}$) denote the class of all (bipartite and planar) graphs G of maximum degree Δ and girth $g(G) \geq |G|^r$. Then for $0 \leq r < 1$,* MAX CUT *restricted to the class $\mathcal{G}_{3,r}$,* DOMINATING SET *and* STEINER TREE *restricted to the class $\mathcal{BPG}_{3,r}$ and* VERTEX FEEDBACK SET *restricted to the class $\mathcal{BPG}_{6,r}$ are* NP*-complete.*

Proof. Restricted to graphs of maximum degree 3 the problems MAX CUT [Yan81], PLANAR DOMINATING SET [KK79] and PLANAR STEINER TREE are NP-complete. As – to our best knowledge – for DOMINATING SET and STEINER TREE these NP-completeness proofs are unpublished, let us briefly sketch the reductions. To show that PLANAR DOMINATING SET is NP-complete, use the reduction [LR79] from PLANAR NODE COVER [GJ77]; as to the restriction of the maximum degree, a local replacement analogously to the corresponding proof [GJ77] for NODE COVER works as follows: replace each node in G of degree $d > 3$ by a circuit of length $3d+1$ and attach a leaf to one of these vertices appropriately. STEINER TREE is NP-complete on unweighted grid graphs [GJ77], which are planar graphs of maximum degree 4. To remove vertices of degree 4 insert $4n$ new vertices on each edge and replace vertices of degree 4 by a 4-cycle. Then, for the graph G' obtained, $St(G,T) = \left\lfloor \frac{1}{4n+1} St(G',T') \right\rfloor$. VERTEX FEEDBACK SET is NP-complete restricted to bipartite planar graphs of maximum degree 6 [KD79].

We set out the reduction for DOMINATING SET only and then indicate how to modify it for the other two problems. So let G be a planar graph of maximum degree $\Delta(G) \leq 3$. Let k be the smallest integer such that $\frac{k}{k+2} \geq r$ and set $t := |G|^k$ if $|G|$ is odd and $t := |G|^k - 1$ otherwise. Insert $3t$ new vertices on each edge and denote the resulting bipartite planar graph by G'. Then $|G'| \leq |G| + \frac{3|G|}{2} 3|G|^k \leq |G|^{k+2}$ for $|G| \geq 5$. Hence $g(G') \geq 3t \geq |G|^k \geq |G'|^{\frac{k}{k+2}} \geq |G'|^r$ so that $G' \in \mathcal{BPG}_{3,r}$. Furthermore $\gamma(G') = \gamma(G) + m(G) \cdot t$ (for "$\geq$" observe that for each edge $e \in E(G)$ a minimum dominating set S in G' contains at least t vertices of the $3t$ vertices inserted on e; on the other hand, without loss of generality, S contains at most t of them). As r is constant the reduction is polynomial time computable.

For MAX CUT resp. VERTEX FEEDBACK SET and STEINER TREE insert $2t$ resp. t vertices on each edge to obtain a graph G' with $mc(G') = mc(G) + m(G) \cdot 2t$ resp. $vfs(G') = vfs(G)$ and $St(G',T) = (t+1) \cdot St(G,T)$. □

For MAX CUT resp. DOMINATING SET and fixed g, the result was already proved in [PT95] resp. [ZZ95]. Analogously, for every fixed r, $0 \leq r < 1$, the problem INDEPENDENT SET (and hence NODE COVER) is NP-complete restricted to the class of all planar graphs G of maximum degree 3 and girth $g(G) \geq |G|^r$ [Mur92].

In Section 5 we will prove the following theorem. Put $\text{llog} n = \log\log n$.

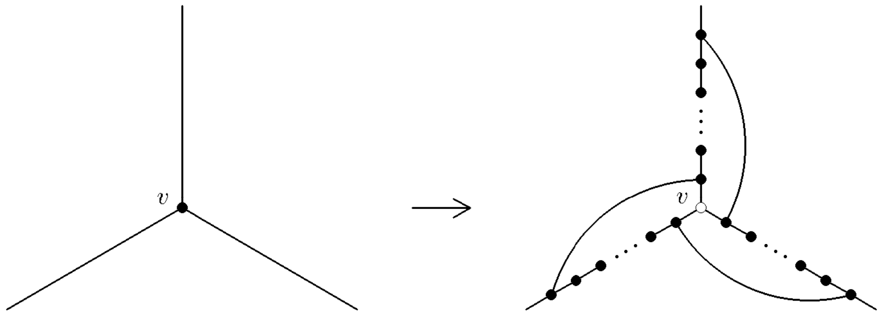
Theorem 2.2 *The problems* MAX CUT, INDEPENDENT SET, NODE COVER, DOMINATING SET, VERTEX FEEDBACK SET *and* STEINER TREE *restricted to the class of graphs G with girth $g(G) \geq \frac{n \text{llog} n}{\log n}$ are solvable in polynomial time.*

3 HAMILTONIAN CIRCUIT

For a graph G, the problem HAMILTONIAN CIRCUIT is to decide whether G contains a circuit visiting each node of G exactly once.

Theorem 3.1 *Let $0 \leq r < 1/2$ be a fixed rational number. Then* HAMILTONIAN CIRCUIT *is* NP*-complete for bipartite planar graphs G of maximum degree 3 and girth $g(G) \geq |G|^r$.*

Proof. The reduction will use the following operation on a graph $G = (V, E)$ which we will refer to as "expanding a node v". Let $t \in \mathbb{N}$ be an integer and $v \in V$ be a node of degree 3. For $i = 1, 2, 3$, insert t new nodes, say $x_1^{(i)}, \ldots, x_t^{(i)}$, on the i-th edge incident to v such that $x_j^{(i)}$ is at distance j from v, $1 \leq j \leq t$, and add the edges $\{x_t^{(1)}, x_1^{(2)}\}$, $\{x_t^{(2)}, x_1^{(3)}\}$, and $\{x_t^{(3)}, x_1^{(1)}\}$, cf. the following figure:



Observe that the resulting graph G_v is Hamiltonian if and only if G is. Furthermore, the graph H which is induced by the new nodes and v has girth $t+2$ and every cycle in G_v which intersects H but which is not completely contained in H has either length at least $t+3$ or uses the node v and at least 4 other nodes of H.

We now reduce from the Hamiltonian Circuit problem on cubic bipartite planar graphs [ANS80]. So let G be a cubic bipartite planar graph. Let k be the smallest integer such that $\frac{k}{2(k+1)} \geq r$ and set $t := |G|^k$ if $|G|$ is even and $t := |G|^k + 1$ otherwise. Expanding every node of G t-times yields a graph G' of order $|G'| \leq |G| \cdot (1 + t \cdot 3t) \leq |G|^{2k+2}$ for $|G|$ large enough. Hence G' is planar and bipartite, has maximum degree 3 and girth

$$g(G') \;\geq\; \min\{(t \cdot 4 + 1) \cdot g(G), t+2\} \;\geq\; t \;\geq\; |G|^k \;\geq\; |G'|^{\frac{k}{2k+2}} \;\geq\; |G'|^r.$$

Furthermore, as r is constant, the reduction is polynomial time computable. □

In fact, the last theorem is essentially tight as the following proposition shows.

Proposition 3.2 *Let $\mathcal{G} := \{G : G$ is a graph with $g(G) > 2\sqrt{|G|}\}$. Then* Hamiltonian Circuit *restricted to the class $\mathcal{G}$ is in* P.

Proof. Let $G \in \mathcal{G}$ and $n := |G|$. We may clearly assume that G has no vertices of degree smaller than 2. Suppose a node v in G has at least two neighbors x and y of degree 2. Then a Hamiltonian circuit must use the edges $\{x, v\}$ and $\{v, y\}$. Therefore we may remove from G the other edges incident to v, the reduced graph being Hamiltonian if and only if G is. This way we may recursively remove edges from G that cannot be contained in a Hamiltonian circuit.

Suppose first that at some step there occurs a vertex of degree less than 2. Then obviously G is not Hamiltonian.

Assume secondly that at some step there are no vertices of degree at least 3 left in the reduced graph. Then G is Hamiltonian if and only if the reduced graph is a $|G|$-cycle.

Suppose finally that the minimum degree in the reduced graph H is at least 2 and that there are still some vertices of degree at least 3 left, but every vertex of degree at least 3 has at least two neighbors of degree at least 3. We will show that this case is impossible. Pick a vertex v of degree at least 3 in H. Let $\Gamma_i(v)$ denote the set of all vertices at distance (exactly) i from v. We claim that

(*) for $1 \leq i \leq \lfloor\sqrt{n}\rfloor$, $\quad |\Gamma_i(v)| \geq 2i + 1$ and
$\Gamma_i(v)$ contains at least two vertices of degree at least 3.

To see this observe that (*) is true for $i = 1$ by the way we chose v and by the definition of H. Now let $1 < i \leq \lfloor\sqrt{n}\rfloor$ and assume that (*) is true for $i-1$. Note that $g(H) \geq g(G) > 2\lfloor\sqrt{n}\rfloor$. As the minimum degree of H is at least 2 and a vertex $w \in \Gamma_{i-1}(v)$ cannot have a neighbor in $\Gamma_{i-1}(v)$ (otherwise there would be a cycle of length at most $2i - 1 \leq 2\lfloor\sqrt{n}\rfloor - 1$)

every vertex $w \in \Gamma_{i-1}(v)$ has at least one neighbor in $\Gamma_i(v)$. Furthermore, for different w's these neighbors must be different because otherwise there would be a cycle in H of length at most $2i \leq 2\lfloor\sqrt{n}\rfloor$. The two vertices of degree at least 3 in $\Gamma_{i-1}(v)$ each have two neighbors in $\Gamma_i(v)$, at least one of them having degree greater than 2. Hence $|\Gamma_i(v)| \geq |\Gamma_{i-1}(v)| + 2$ and (*) is proven.

Now, by (*), the number of vertices at distance $0, \ldots, \lfloor\sqrt{n}\rfloor$ from v is at least

$$\sum_{i=0}^{\lfloor\sqrt{n}\rfloor} (2i+1) \;=\; (\lfloor\sqrt{n}\rfloor + 1)^2 \;>\; n,$$

a contradiction. □

4 Chromatic Index

A k-edge-coloring of a graph $G = (V, E)$ is a function $c : E \to \{1, \ldots, k\}$ such that incident edges receive different colors, that is, for all edges $e \neq f$ in G, $e \cap f \neq \emptyset \Rightarrow c(e) \neq c(f)$. The minimum number k of colors such that there exists a k-edge-coloring of G is called the *chromatic index* of G, denoted by $\chi'(G)$. By a theorem of Vizing $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$. However, the problem Chromatic Index, which is to decide whether the chromatic index of a graph G is $\Delta(G)$ (a so-called "class one" graph) or $\Delta(G) + 1$ (a "class two" graph) is NP-complete, even for r-regular graphs for any fixed $r \geq 3$ [LG83].

We begin by constructing r-regular graphs of large girth which are class one. These graphs will later be used as gadgets in our NP-completeness proof. By a classical theorem of König bipartite graphs are class one.

Theorem 4.1 *Let $g \geq 3$ and $r \geq 3$ be integers. Then an r-regular graph of girth at least g on*

$$n := n(g, r) := \begin{cases} 2\,\frac{(r-1)^{g-1}-1}{r-2} & \text{if } g \text{ is odd} \\ 4\,\frac{(r-1)^{g-2}-1}{r-2} & \text{if } g \text{ is even} \end{cases}$$

vertices can be constructed in time polynomial in n. Furthermore, in case g is even the graph constructed is bipartite.

Proof. We will show how the non-constructive proof of [Bol78, Theorem III.1.4] can be turned into an algorithm.

The case g even is derived from the case g odd as follows, see [Bol78, Theorem III.1.3]. Let $G = (V, E)$ be an r-regular graph of girth $g-1$ and order $n(g-1, r)$. Then an r-regular bipartite graph $G' = (V', V'', E')$ of girth at least g is defined by letting V' and V'' be disjoint copies of V and $E' := \{\{x', y''\} : x' \in V', y'' \in V'', \{x, y\} \in E\}$.

Consider now the case g odd. Let V be a vertex set of size n; we will see in a moment that n is an integer. The following procedure constructs an r-regular graph $G = (V, E)$ on V of girth at least g. By $d(v)$ we will denote the degree of a vertex $v \in V$ with respect to the current edge-set E and $dist(v, w)$ the distance between vertices v and w which is defined to be ∞ if there is no path from v to w. For a node $x \in V$ and an integer $k \in \mathbb{N}_0$, $\mathcal{B}_k(x) := \{v \in V |\; dist(v, x) \leq k\}$ is the ball of radius k around x, $\mathcal{S}_k(x) := \{v \in V |\; dist(v, x) = k\}$ the sphere at radius k around x.

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E := ∅;
REPEAT
  WHILE there exist x1, x2 ∈ V with d(x1) < r and d(x2) < r
        and dist(x1, x2) ≥ g − 1 DO   E := E ∪ {{x1, x2}};
  IF |E| < nr/2 THEN BEGIN
    choose x1, x2 ∈ V with d(x1) < r and d(x2) < r;
    choose an edge {y1, y2} ∈ E such that
           y1 ∈ V \ (B_{g-2}(x1) ∪ B_{g-2}(x2)) and
           y2 ∈ V \ (B_{g-2}(x1) ∩ B_{g-2}(x2));
    IF dist(y2, x1) ≥ g − 1 THEN
      E := E\{{y1, y2}} ∪ {{y2, x1}, {y1, x2}};
    ELSE
      E := E\{{y1, y2}} ∪ {{y2, x2}, {y1, x1}};
  END; {IF |E| < nr/2}
UNTIL |E| = nr/2;
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The proof of the correctness of this procedure is essentially that of [Bol78, Theorem III.1.4]. For the sake of completeness we will provide it here, too.

We will show that the choice of $\{y_1, y_2\} \in E$ as specified can always be made and that at all times the following invariant holds for $G = (V, E)$

(i) $\Delta(G) \leq r$, and

(ii) $g(G) \geq g$.

Observe that (i) and (ii) hold in the beginning and that these conditions are not violated during the WHILE-loop. As to the choice of $\{y_1, y_2\}$ observe that for $i = 1, 2$

$$\mathcal{B}_{g-2}(x_i) \;\leq\; \sum_{j=0}^{g-2}(r-1)^j \;=\; \frac{(r-1)^{g-1}-1}{r-2} \;=\; n/2.$$

The WHILE-loop guarantees that every vertex of degree at most $r-1$ is in $\mathcal{B}_{g-2}(x_1) \cap \mathcal{B}_{g-2}(x_2)$, in particular so are x_1 and x_2. Trivially, $\mathcal{S}_{g-2}(x_1) \cap \mathcal{S}_{g-2}(x_2) \subseteq \mathcal{B}_{g-2}(x_1) \cap \mathcal{B}_{g-2}(x_2)$ so that for $R := V \setminus (\mathcal{B}_{g-2}(x_1) \cup \mathcal{B}_{g-2}(x_2))$

$$|R| \;=\; n - |\mathcal{B}_{g-2}(x_1)| - |\mathcal{B}_{g-2}(x_2)| + |\mathcal{B}_{g-2}(x_1) \cap \mathcal{B}_{g-2}(x_2)| \;\geq\; |\mathcal{S}_{g-2}(x_1) \cap \mathcal{S}_{g-2}(x_2)| \;+\; 2.$$

Hence $R \neq \emptyset$ and any vertex in R satisfies the requirement for y_1 in the algorithm. Neighbors of vertices in R have distance at least $g-2$ to x_1 as well as to x_2. As all vertices in R have degree r the above inequality shows that not all neighbors of vertices in R can have distance $g-2$ to x_1 as well as to x_2 since then there would be a vertex of degree larger then r in $\mathcal{S}_{g-2}(x_1) \cap \mathcal{S}_{g-2}(x_2)$, contradicting invariant (1). Therefore, an edge $\{y_1, y_2\} \in E$ as specified in the algorithm exists.

We now argue that the update of E with respect to $\{y_1, y_2\}$ does not violate invariant (ii). We only consider the case $dist(y_2, x_1) \geq g-1$, the other case then follows by symmetry. Obviously, none of the edges $\{y_2, x_1\}$ and $\{y_1, x_2\}$ introduces a cycle of length less than g in $G - \{y_1, y_2\}$ by itself. Suppose there is a cycle in $G - \{y_1, y_2\}$ of length less than g containing both edges. Then there would either be a $y_1 - y_2-$ or a $y_1 - x_1-$path of length at most $g-3$ in $G - \{y_1, y_2\}$ which is impossible.

Finally, using a polynomial time shortest path algorithm the procedure is obviously computable in time polynomial in n since each iteration of the REPEAT-loop adds at least one to $|E|$. □

Corollary 4.2 *For integers $r \geq 3$ and $g \geq 3$, there exists a bipartite r-regular graph of girth at least g on at most $n = 4(r-1)^{g-1}$ vertices. Furthermore, such a graph can be constructed in time polynomial in n.* □

Let G be an r-regular graph and H be a bipartite r-regular graph. Fix an arbitrary edge $h = \{x, y\}$ of H. For an edge $e = \{u, v\}$ of G consider the following operation on G which we will refer to as $e \to H$: remove the edge e from G, take a copy of H, remove the edge h from H, and add the two edges $\{u, x\}$ and $\{v, y\}$ to the disjoint union $(G-e) \cup (H-h)$. Let the resulting graph be called $G[e \to H]$. Observe first that $G[e \to H]$ is again r-regular. Furthermore, we have:

Lemma 4.3 *G is r-edge-colorable if and only if $G[e \to H]$ is.*

Proof. As H is r-regular and class one, i.e. r-edge-colorable, the only-if part is obvious. For the reverse implication, let us first compute the chromatic index of the graph H' which arises from H by subdividing the edge $\{x, y\}$ by one new node. The edge set of H decomposes into r perfect matchings. As H' has only one vertex more than H the size of a maximum matching in H' is the same as in H. Hence, since H' contains exactly one edge more than H, it takes at least $r+1$ matchings to cover its edges.

Now, consider an r-edge-coloring of $G[e \to H]$. The edges $\{u, x\}$ and $\{v, y\}$ must have received the same color, say b, because otherwise the graph H' would be r-edge-colorable. Hence, by assigning the color b to the edge $\{u, v\}$, the r-edge-coloring of $G[e \to H]$ induces an r-edge-coloring of G. □

Theorem 4.4 *Let an integer $r \geq 3$ and a rational number c, $0 \leq c < 1$, be fixed. Then the problem* Chromatic Index *is* NP*-complete for r-regular graphs G of girth $g(G) \geq c\frac{\log|G|}{\log(r-1)}$.*

Proof. We reduce from the NP-complete problem Chromatic Index on r-regular graphs, so let $G = (V, E)$ be an r-regular graph on $n := |G|$ vertices.

For $g := \left\lceil \frac{2c}{1-c}\frac{\log n}{\log(r-1)} \right\rceil$, construct a bipartite r-regular graph H of girth at least g as in Corollary 4.2. Fix an edge $h = \{x, y\}$ of H arbitrarily. Now, for all edges $e \in E$, successively apply the operation $e \to H$. By Lemma 4.3 the resulting r-regular graph G' is r-edge-colorable if and only if G is. By Corollary 4.2 H has order $|H| \leq 4(r-1)^{g-1} < 4n^{\frac{2c}{1-c}}$ and thus G' has order

$$|G'| \;\leq\; n + \frac{nr}{2}|H| \;\leq\; n^2 n^{\frac{2c}{1-c}} \;=\; n^{\frac{2}{1-c}},$$

for $n \geq 4r$ (for a smaller graph G, $\chi'(G)$ can be computed in constant time by complete enumeration). As the girth of G' obviously is at least as large as the girth of H, we finally get

$$g(G') \;\geq\; g \;\geq\; \frac{2c}{1-c}\frac{\log n}{\log(r-1)} \;\geq\; c\frac{\log|G'|}{\log(r-1)}.$$ □

It is easy to see that with respect to the girth Theorem 4.4 is best possible up to a constant factor since counting the nodes at distance at most $\lfloor \frac{g-1}{2} \rfloor$ of any particular node yields $g(G) \leq 2\frac{\log|G|}{\log(r-1)} + 2$ for any r-regular graph G.

5 The use of small feedback sets

Analogously to a vertex feedback set (see Section 2) an *edge feedback set* in a graph $G = (V, E)$ is defined to be an edge set $A \subseteq E$ such that the graph $(V, E \backslash A)$ is a forest. For a graph G, denote by $H(G)$ the graph obtained from G by recursively removing vertices of degree 0 and 1 from G.

Theorem 5.1 *Let $G = (V, E)$ be a connected graph with girth $g(G) \geq \frac{n \operatorname{llog} n}{\log n}$, where $n = |G|$. Then the set $F = \{v \in V : d_H(v) \geq 3\}$ is incident to at most $\mathcal{O}(\log n)$ edges from $H := H(G)$. Hence G contains an edge feedback set and a vertex feedback set of size $\mathcal{O}(\log n)$ which can be computed in polynomial time.*

Proof. We need only prove the statement about F. As G is connected $H = H(G)$ is also connected. If H is empty or just a cycle then the statement is evident. If this is not the case then $F = \{v \in V : d_H(v) \geq 3\} \neq \emptyset$.

Let $m = |E(H)|$. Then $|F| \leq 2(m - |H|)$ and F is incident to at most

$$\sum_{v \in F} d_H(v) \;=\; 2m - 2(|H| - |F|) \;=\; 2|F| + 2(m - |H|) \;\leq\; 6(m - |H|)$$

edges in H. Hence we only need to show that $m - |H| = \mathcal{O}(\log |H|) = \mathcal{O}(\log n)$.

If $m > 2|H|$ then by [Bol78, Theorem III.3.7(a)] $g(G) = g(H) \leq 2 \log |H| + 2$ which is impossible if n is large enough. Hence $m \leq 2|H|$.

Assume that $m - |H| \geq 7 \log |H|$. By [Bol78, Theorem III.3.6] H contains a collection of at least k edge-disjoint cycles where k has to satisfy $m - |H| \leq 2k(\log k + \operatorname{llog} k + 2)$. Solving for k gives $\frac{m - |H|}{3 \log(m - |H|)}$ edge-disjoint cycles provided $|H|$ is large enough. The sum of the lengths of these cycles is at most $m \leq 2|H|$. Thus there is one cycle of length at most

$$2|H| \frac{3 \log(m - |H|)}{m - |H|} \;\leq\; \frac{6|H| \log(7 \log |H|)}{7 \log |H|} \;<\; \frac{n \operatorname{llog} n}{\log n},$$

provided n is large enough yielding a contradiction. □

The following proposition proves Theorem 2.2 for the problems Max Cut, Independent Set and Node Cover.

Proposition 5.2 *The problems* Max Cut, Independent Set *and* Node Cover *are solvable in polynomial time given a vertex feedback set of size $\mathcal{O}(\log |G|)$.*

Proof. We only give an algorithm for the problem Max Cut as the other problems can be treated similarly.

Let $G = (V, E)$ be a graph and suppose $\emptyset \neq F \subseteq V$. For a partition of F into $F = X_F \dot{\cup} Y_F$ let $mc(X_F)$ denote the maximum number $|E(X, Y)|$ of cut edges in a partition $V = X \dot{\cup} Y$ of V which extends $F = X_F \dot{\cup} Y_F$, i.e. for which $X_F \subseteq X$ and $Y_F \subseteq Y$. Then $mc(G)$ is simply max $\{mc(X_F) : X_F \subseteq F\}$.

We will show below that $mc(X_F)$ can be computed in polynomial time in case F is chosen to be a vertex feedback set for G. Now, since we are given a vertex feedback set F of size $|F| = \mathcal{O}(\log |G|)$, $mc(G)$ may be determined in polynomial time by enumerating over all $X_F \subseteq F$.

For a vertex feedback set F and an $X_F \subset F$, $mc(X_F)$ can be computed in polynomial time as follows. Let $G[V \backslash F]$ consist of the trees $T^1, \ldots, T^k$ and denote by E^i the set of edges incident with the vertices of T^i. Observe that E can be written as the disjoint union

$E = E(X_F, Y_F)\dot\cup E^1 \dot\cup \ldots \dot\cup E^k$ and that the number of cut edges in E^i depends solely on the assignment of the vertices in T^i to X and Y.

The maximum number of cut edges in E^i for a tree T^i is now computed in a dynamic programming fashion. Choose a root w_i of T^i. Denote by T^i_v the subtree of T^i rooted at v, i.e. those vertices u of T^i for which the unique $u - w_i$-path in T^i contains v.

For a vertex $v \in T_i$, let the two functions $f^x(v)$ resp. $f^y(v)$ denote the maximum number of edges in a cut in G extending $(X_F \cup \{v\}, Y_F)$ resp. $(X_F, Y_F \cup \{v\})$ which belong to $E(T^i_v) \cup E(V(T^i_v), F)$. Hence, the maximum number of cut edges in E^i is $\max\{f^y(w_i), f^x(w_i)\}$.

Writing $\Gamma^-(v)$ for the set of neighbors of v in T^i_v we have the following recursions

$$\begin{aligned} f^x(v) &= |\Gamma(v) \cap Y_F| + \sum_{u \in \Gamma^-(v)} \max\{f^x(u), 1 + f^y(u)\} \\ f^y(v) &= |\Gamma(v) \cap X_F| + \sum_{u \in \Gamma^-(v)} \max\{f^y(u), 1 + f^x(u)\}, \end{aligned}$$

at the leaves of T^i using the convention that the sum over the empty set is zero. Thus the values of f^x and f^y for all $v \in T^i$ can be computed in the reverse order of the order in which a breadth first search starting from root w_i visits the nodes of T^i.

The value $mc(X_F)$ is finally determined according to

$$mc(X_F) = |E(X_F, Y_F)| + \sum_{i=1}^{k} \max\{f^y(w_i), f^x(w_i)\}. \qquad \square$$

Now we are going to prove Theorem 2.2 for the problem Dominating Set by making use of the existence of a small edge feedback set. To this end let us first consider the problem Dominating Set With Preassignments which is defined as follows. Let $G = (V, E)$ be a graph and let a function $status : V \to \{0, 1, 2\}$ be given. The meaning of the function *status* is that vertices with status 1 are preassigned to be in the dominating set, vertices with status 2 are not in the preassigned dominating set but need not be dominated and vertices with status 0 still need to be dominated. Hence the problem is to find a minimum size $D \subseteq V$ such that

- $D \supseteq \{v \in V : status(v) = 1\}$ and
- $\{v \in V : status(v) = 0\} \subseteq D \cup \Gamma(D)$,

By abuse of notation we will say that such a set D dominates the vertices of G. The following lemma is a generalization of [CGH75].

Lemma 5.3 *The problem* Dominating Set With Preassignments *is solvable in polynomial time on trees.*

Proof. Let a tree $T = (V, E)$ and a function $status : V \to \{0, 1, 2\}$ be given. We will use the same terminology as in the proof of Proposition 5.2. Fix a root w of T and consider the following three functions for a vertex $v \in V$:

$$\begin{aligned} f^{in}(v) &:= \min\{|D| : D \subseteq V(T_v), v \in D \text{ and } D \text{ dominates all vertices in } V(T_v)\}, \\ f^{out}(v) &:= \min\{|D| : D \subseteq V(T_v), v \notin D \text{ and } D \text{ dominates all vertices in } V(T_v)\}, \\ f^{ex}(v) &:= \min\{|D| : D \subseteq V(T_v) \text{ and } D \text{ dominates all vertices in } V(T_v) \\ &\qquad\qquad \text{except possibly } v \text{ itself}\} \end{aligned}$$

where "dominates" means "dominates with respect to the preassignment function *status*". Then the number we are heading at is just $\min\{f^{in}(w), f^{out}(w)\}$.

The three functions satisfy the following recursions for a vertex $v \in V$:

$$
\begin{aligned}
f^{in}(v) &= 1 + \sum_{u \in \Gamma^-(v)} f^{ex}(u), \\
f^{out}(v) &= \begin{cases} \min_{w \in \Gamma^-(v)} \{ f^{in}(w) + \sum_{u \in \Gamma^-(v) \setminus \{w\}} \min\{f^{in}(u), f^{out}(u)\}\} & : \ status(v) = 0, \\ \infty & : \ status(v) = 1, \\ \sum_{u \in \Gamma^-(v)} \min\{f^{in}(u), f^{out}(u)\} & : \ status(v) = 2, \end{cases} \\
f^{ex}(v) &= \begin{cases} f^{in}(v) & : \ status(v) = 1, \\ \min\left\{ f^{in}(v), \sum_{u \in \Gamma^-(v)} \min\{f^{in}(u), f^{out}(u)\} \right\} & : \ status(v) \in \{0, 2\}. \end{cases}
\end{aligned}
$$

It is easy to check that the formulae give the right values in particular for the leaves of T using the convention that the sum over the empty set is 0 and the minimum over the empty set is infinity. Hence, the values of f^{in}, f^{out} and f^{ex} can be computed by dynamic programming in the reverse order of the order in which a breadth first search starting from root w visits the nodes of T yielding the size $\min\{f^{in}(w), f^{out}(w)\}$ of a minimum dominating set for T at the root w.

Furthermore, it should be clear that a minimum dominating set for T can now be computed by another breadth first search from w. □

Proposition 5.4 *The problem* Dominating Set *is solvable in polynomial time given an edge feedback set of size at most* $\mathcal{O}(\log n)$.

Proof. Let A be an edge feedback set for G of size at most $\mathcal{O}(\log n)$. Without loss of generality G is connected and A is an inclusionwise minimal edge feedback set for G. Then $T = (V, E \backslash A)$ is a tree. Let U be the set of vertices of G that are incident with an edge in A. Then also $|U| = \mathcal{O}(\log n)$. A minimum dominating set for G can now be computed by enumerating over all $D_U \subseteq U$, solving the problem Dominating Set With Preassignments for T and

$$
status(v) := \begin{cases} 1 & \text{if } v \in D_U, \\ 2 & \text{if } v \in (U \setminus D_U) \text{ and } v \in \Gamma(D_U), \\ 0 & \text{otherwise} \end{cases}
$$

and taking the smallest dominating set for T obtained this way. □

Now we turn to a third class of problems where we, instead of using just any given small feedback set, have to make use of the structure of the feedback set guaranteed by Theorem 5.1.

A minimal vertex feedback set in a connected graph G with maximum degree $\Delta(G) \geq 3$ contains without loss of generality only vertices from the set $F = \{v \in V : d_H(v) \geq 3\}$, $H = H(G)$. Hence, according to Theorem 5.1 enumerating over all subsets of F yields the following result.

Proposition 5.5 *The problem* Vertex Feedback Set *restricted to the class of graphs G with girth $g(G) \geq \frac{n\text{llog}n}{\log n}$ is solvable in polynomial time.* □

Proposition 5.6 *The* Weighted Steiner Tree *problem restricted to the class of graphs G with girth $g(G) \geq \frac{n\text{llog}n}{\log n}$ is solvable in polynomial time.*

Proof. Let (G, T, w), $G = (V, E)$, be an instance of the WEIGHTED STEINER TREE problem, where $T \subseteq V$ is the set of terminals and $w : E \to \mathbb{Q}^+$ the weight function. We may clearly assume that G is connected. Moreover, we may assume that $G = H(G)$ because suppose G contains a vertex v of degree 1. If v is not a terminal then we may just remove v from G. Otherwise, if v is a terminal, we may define a new equivalent instance by letting the neighbor of v be a terminal and removing v. We finally can assume that there are no non-terminal vertices of degree 2 because we may remove such a vertex and merge the two adjacent edges into a single edge its weight being the sum of the weights of the two old edges.

Let $F = \{v \in V : d(v) \geq 3\}$. G can be written as a union of paths $P^1, \ldots, P^k$ with first and last vertices in F and whose interior vertices lie in $V \backslash F$. By Theorem 5.1, $k = \mathcal{O}(\log n)$. For $i = 1, \ldots, k$, let $P^i = v_1^i, v_2^i, \ldots, v_{j(i)}^i$. Let furthermore $\{v_{\ell(i)}^i, v_{\ell(i)+1}^i\}$ be the longest edge in P^i, i.e. the edge with the maximum weight. Then it is easy to see that a minimum Steiner tree intersects P^i either

(1) in all edges or

(2) in all edges except $\{v_1^i, v_2^i\}$ or

(3) in all edges except $\{v_{j(i)-1}^i, v_{j(i)}^i\}$ or

(4) in all edges except $\{v_{\ell(i)}^i, v_{\ell(i)+1}^i\}$,

where for some i some of the cases may coincide. For a function $case : \{1, \ldots, k\} \to \{1, 2, 3, 4\}$, define a subgraph G_{case} of G such that G_{case} intersects each P^i exactly in those edges specified by $case(i)$. Let $w(G_{case})$ be the sum of the lengths of the edges of G_{case}, then the length of a minimum Steiner tree for (G, T) is

$$St(G, T) = \min \{w(G_{case}) \mid \text{case} : \{1, \ldots, k\} \to \{1, 2, 3, 4\} \text{ and } G_{case} \text{ is a tree covering all vertices from } T\}.$$

As $k = \mathcal{O}(\log n)$ we can enumerate all cases in polynomial time. □

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