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Uniquely Colourable Graphs and the Hardness of Colouring Graphs of Large Girth

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Abstract. For any integer k, we prove the existence of a uniquely k-colourable graph of girth at least g on at most $k^{12(g+1)}$ vertices whose maximal degree is at most $5k^{13}$. From this we deduce that, unless NP=RP, no polynomial time algorithm for k-COLOURABILITY on graphs G of girth $g(G) \geq \frac{\log |G|}{13 \log k}$ and maximum degree $\Delta(G) \leq 6k^{13}$ can exist. We also study several related problems.

1. Introduction and Results

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The order of G, i.e. the number of its vertices, is denoted by $|G|$, the number of its edges by $m(G)$. The degree of a vertex $v \in V$ is denoted by $d(v)$, the maximum degree of a vertex in G by $\Delta(G)$. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G and can easily be computed in polynomial time using breadth-first-search. Let k be an integer, or more generally, an integer valued function. A graph $G = (V, E)$ is called *k-colourable* if and only if there is a function $c : V \to \{1, \ldots, k\}$, where $k = k(|G|)$, such that $c(u) \neq c(v)$ for all edges $\{u, v\} \in E$. Such a function c is called a *k-colouring* of G. Note that c partitions V into independent sets $c^{-1}(i)$, $i = 1, \ldots, k$, the so-called *colour classes* of c. The minimum number k such that a graph G is k-colourable is called the *chromatic number* $\chi(G)$ of G. A graph $G = (V, E)$ is said to be *uniquely k-colourable* if and only if G is k-colourable and every k-colouring of G induces the same k-partition of V. Necessarily, $\chi(G) = k$ for any uniquely k-colourable graph G. For an integer-valued function k, k-COLOURABILITY is the decision problem "given a graph G, is $\chi(G) \leq k(|G|)$?". For an introduction to the theory of computational complexity we refer the reader to [10, 21].

Uniquely colourable graphs of large girth. Graphs of large girth are rather sparse as the maximum number of edges in an n-vertex graph of girth $g \geq 2h+2$, $h \in \mathbb{N}$, is at most $\left(\frac{n}{2}\right)^{1+1/h} + 2^h \cdot \left(\frac{n}{2}\right)^{1-1/h}$ [9]. The problem to decide whether graphs of large girth exist which have large chromatic number and if so how to construct such graphs dates back to 1947 and has received considerable attention in graph theory since then, cf. [13, section 1.5].

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The existence of uniquely k-colourable graphs of arbitrary large girth was already proven by Bollobás and Sauer [5]. However, they give no bound on the order of such a graph. Analyzing their construction one can see that the graph they construct is at least of order g^g.

We use a similar probabilistic argument but remove vertices instead of edges from a random balanced k-partite graph, which yields

Theorem 1.1. *Let $k \geq 2$ be an integer. For every integer $N_1 \geq k$ there exists a graph G on N vertices with $N_1/2 \leq N \leq N_1$ which is uniquely k-colourable and has girth $g(G) \geq \lfloor \frac{\log N_1}{12 \log k} \rfloor$ and maximum degree $\Delta(G) \leq 5k^{13}$.*

Furthermore there is a randomized algorithm which outputs a graph G on N vertices together with a k-colouring of G such that $g(G) \geq \lfloor \frac{\log N_1}{12 \log k} \rfloor$, $\Delta(G) \leq 5k^{13}$ and with probability at least $1/2$ the graph G is uniquely k-colourable.

Theorem 1.1 immediately implies the following result.

Corollary 1.1. *For fixed integers k and g, there exists a uniquely k-colourable graph of girth at least g on at most $k^{12(g+1)}$ vertices whose maximum degree is at most $5k^{13}$.* □

Here the bound on the order is best possible up to the constant in the exponent, which is by [3, Theorem V.4.2] at least $1/2$. The constant in the exponent can be improved slightly, especially for larger k one can easily get a smaller constant. In order to keep the proofs simple we did not optimize on this constant. Furthermore, even when the condition of unique colourability is omitted, i.e. when asking for the minimal order $n(k,g)$ of a graph with chromatic number k and girth at least g, the constant in the exponent is not known. In [4] it is shown that $n(k,g) \leq \lceil (6k \log k)^{g+1} \rceil$. Lubotzky, Phillips und Sarnak [18] succeeded in constructing such graphs of order $n \leq k^{3g}$ explicitly. Recent results of Kim [17] and Johansson [14] on the other hand give a lower bound of $\Omega(k \log k)$ on the maximum degree of such graphs when $g \geq 5$ resp. $g \geq 4$.

Hardness results. That bounding the maximum degree in optimization problems on graphs often does not affect their hardness is well documented, see e.g. [16]. The girth as another graph parameter controlling the sparseness of a graph is less well studied in this context. As graphs of large girth look like trees locally, one might think that an NP-complete problem might be easier to solve on graphs of large girth. In spite of being solvable in linear time on trees, however, several well-known NP-hard problems on graphs remain NP-hard on graphs with large girth. This was first observed for the problem Independent Set by Murphy [20] who showed that this problem remains NP-complete even when restricted to the set of all graphs G with girth $g(G) \geq |G|^r$ where $0 \leq r < 1$ is a fixed real number. In a companion paper [7] we study the complexity of several graph theoretic problems like Hamiltonian Circuit, Chromatic Index and Steiner Tree on graphs of large girth and bounded maximum degree.

In this note we study the complexity of graph colouring problems on graphs of large

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girth and bounded maximum degree. Our reductions rely heavily on properties of uniquely colourable graphs whose existence is guaranteed by Theorem 1.1.

Theorem 1.2. *Let $k \geq 3$ be a fixed integer. Unless* NP=RP, *there exists no polynomial time algorithm for k*-COLOURABILITY *on the class of graphs G of girth $g(G) \geq \lfloor \frac{\log|G|}{13 \log k} \rfloor$ and maximum degree $\Delta(G) \leq 6k^{13}$.*

The bound on the girth is best possible up to a constant factor as a graph G with chromatic number k has girth at most $2\frac{\log|G|}{\log(k-2)} + 1$, cf. [3, Theorem V.4.2, p.258].

For a finite class of graphs $\mathcal{G}$, we denote by $\mathcal{F}orb\ (\mathcal{G})$ the class of graphs containing no graph from $\mathcal{G}$ as a (weak) subgraph. In abuse of notation, we will write $\mathcal{F}orb\ (G)$ for $\mathcal{F}orb\ (\{G\})$ in case of a single forbidden graph G. For fixed g, the proof of Theorem 1.2 implies:

Corollary 1.2. *(a) Let $k \geq 3$ and g be fixed integers. Then the problem k*-COLOURABILITY *is* NP*-complete on the class of graphs of girth at least g and of maximum degree at most $6k^{13}$.*
(b) Let $k \geq 3$ be an integer. Let $\mathcal{G}$ be a finite class of graphs such that every graph from $\mathcal{G}$ either contains a cycle or a vertex of degree more than $6k^{13}$. Then the problem k-COLOURABILITY *is* NP*-complete on the class $\mathcal{F}orb\ (\mathcal{G})$.*

The NP-completeness of 6-COLOURABILITY on the class of graphs of girth at least g, g a fixed integer, was proven by JENSEN and TOFT [13, section 10.3] using HAJÓS' construction and in fact their proof works for any fixed k and g. However, their method does not give any bound on the maximum degree and, moreover, their reduction is not polynomial for k and g that grow with $|G|$. In Section 3 we will also prove a stronger form of Theorem 1.2 where k may also be an integer-valued, non-decreasing function.

Note that, trivially, k-COLOURABILITY is polynomially solvable on bipartite graphs and that, asymptotically, almost all triangle-free graphs (i.e. graphs with girth $g(G) \geq 4$) are bipartite [8]. In contrast, Corollary 1.2(a) shows that k-COLOURABILITY is NP-complete already on the class of triangle-free graphs. Corollary 1.2(b) contrasts with the existence of an algorithm that, for fixed $\ell \in \mathbb{N}$, colours the graphs in $\mathcal{F}orb\ (K_\ell)$ in linear expected time [22].

In the case $k = 3$ and $g = 4$ we can improve on Corollary 1.2(a) with respect to the degree, using a modification of a well-known uniquely 3-colourable triangle-free graph as a gadget. This result has been proved independently by Maffray and Preissmann [19] using a similar reduction; thus we omit our proof here.

Theorem 1.3. 3-COLOURABILITY *is* NP*-complete for the class of triangle-free graphs G of maximum degree $\Delta(G) \leq 4$.* □

The bound on $\Delta(G)$ is of course best possible as by BROOKS' theorem a triangle-free

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graph G with $\Delta(G) \leq 3$ is 3-colourable. Observe also that 3-COLOURABILITY is in P for planar triangle-free graphs by GRÖTZSCH's theorem [11].

Another problem we address is the complexity of "colouring with many colours" as measured by the maximum degree of a graph. The proof of BROOKS' theorem yields a linear time algorithm to k-colour a graph of maximum degree k, $k \geq 3$, that does not contain a K_{k+1} as a subgraph. It is however an open problem whether for $k \geq 9$, there exists a graph of maximum degree k not containing a K_k which has chromatic number k or, on the other hand, whether BROOKS' theorem can be strengthened, cf. [13, sections 4.7, 4.8]. By a recent related result of REED [23] the chromatic number of a graph can be bounded by a convex combination of its clique number and $\Delta + 1$, for sufficiently large Δ. We show the following.

Theorem 1.4. *Let $k \geq 3$ be a fixed integer. Then k-*COLOURABILITY *is* NP*-complete on the class of all graphs G of maximum degree $\Delta(G) \leq k + \lceil\sqrt{k}\rceil - 1$.*

Observe that this result is best possible for $k \leq 4$. Furthermore, for graphs of girth at least 5, the maximum degree must be superlinear in the chromatic number [17].

Outline of the paper. In section 2 we prove theorem 1.1. In section 3 we therefrom derive our hardness results.

2. Uniquely Colourable Graphs of Large Girth

In this section we prove Theorem 1.1. We use a construction similar as in [5], however, instead of removing edges from a random k-partite graph we will remove vertices.

First we define a class of k-colourable graphs. In Lemma 2.1 it is then shown that many of these graphs have some useful properties. It is then shown how to construct from a graph with these properties a uniquely k-colourable graph with large girth.

For integers n and k, fix k disjoint sets $V_1, \ldots, V_k$ with $|V_i| = n$ for all $i = 1, \ldots, k$. Let $\mathcal{G}(n, k)$ be the class of labelled k-partite graphs on the vertex set $V_1 \cup V_2 \cup \ldots \cup V_k$, i.e. there may only be edges between different V_is whereas the V_is are stable sets.

Let $p = p(n)$ with $0 \leq p \leq 1$. Define a probability distribution on $\mathcal{G}(n, k)$ by assigning to a graph $G \in \mathcal{G}(n, k)$ with e edges the probability $p^e(1-p)^{n^2k(k-1)/2-e}$. Equivalently, we may construct an element $G \in \mathcal{G}(n, k)$ by picking for each $i, j \in [k]$ with $i \neq j$ and for all vertices $v \in V_i$, $w \in V_j$ the edge $\{v, w\}$ with probability p. We finally denote by $\mathcal{G}(n, k)_p$ the probability space defined in this way.

Now we define a subclass of $\mathcal{G}(n, k)$ whose graphs have some suitable properties.

Definition. Let $\mathcal{F}(n, k)$ be the class of graphs $G \in \mathcal{G}(n, k)$ having the following properties.

(i) For any $i, j \in [k]$ with $i \neq j$ and any $U \subset V_i$, $W \subset V_j$ of size

$$|U| = \lceil k^{-3}n/2\rceil \qquad \text{and} \qquad |W| = \lceil (k-1)n/k\rceil$$

there are at least $k^7n/4$ edges between U and W in G.

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(ii) For any $i, j \in [k]$ with $i \neq j$ and any $U \subset V_i$, $W \subset V_j$ of size $|U| = |W| = \lceil \frac{n}{40k} \rceil$ there is at least one edge between U and W in G.
(iii) For any $i \in [k]$ and any $U \subset V_i$, $W \subset \bigcup_{j \neq i} V_j$ of size $1 \leq |U| = |W| \leq n/40$ there are less than $|W|k^{10}/2$ edges between U and W in G.
(iv) Let $g := \lfloor \frac{1}{11} \frac{\log n}{\log k} \rfloor$ and define $C := \{v : v$ is a vertex contained in a cycle in G of length $3, \ldots, g-1\}$. Then $|C| \leq \frac{n}{4k}$.
(v) Let $Y := \{v : v$ has degree in G larger than $5k^{13}\}$. Then $|Y| \leq \frac{n}{4k} - 1$.

For the proof of the next lemma we need a special case of the Chernoff-bounds, cf. [1]. Let X be a random variable with binomial distribution $\mathcal{B}(N,p)$. Then we have

$$Prob(X < (5/6)Np) \leq \exp(-Np/72). \qquad (1)$$

Lemma 2.1. *Let $k \geq 3$ and $n \geq k^{10}$ be integers. Define $p = k^{10}n^{-1}$. Then an element $G \in \mathcal{G}(n,k)_p$ is contained in $\mathcal{F}(n,k)$ with probability at least $1/2$.*

Proof. We are going to show that an element $G \in \mathcal{G}(n,k)_p$ has each of the properties (i)-(v) with probability at least $9/10$.
Ad (i). Let for some $i \neq j$, $U \subset V_i$ with $|U| = \lceil k^{-3}n/2 \rceil$ and $W \subset V_j$ with $|W| = \lceil (k-1)n/k \rceil$ be given. Let $N = \lceil k^{-3}n/2 \rceil \cdot \lceil (k-1)n/k \rceil$. Then $(5/6)Np \geq k^7 n/4$. By (1) the probability of the event that U and W span less than $k^7n/4$ edges is at most $\exp(-Np/72) \leq \exp(-k^7n/216)$.

There are at most k^2 choices for i and j and at most 2^{2n} ways to choose U and W. Hence the probability that the statement of (i) does not hold is at most

$$k^2 2^{2n} \exp(-k^7n/216) \;\leq\; k^2 \exp((2 - k^7/216)n) \;\leq\; k^2 \exp(-2n) \;\leq\; 1/10$$

as $k^7 \geq 3^7 > 4 \cdot 216$ and $2n \geq k^{10} \geq 2 \ln k + \ln 10$.
Ad (ii). Let for some $i \neq j$, $U \subset V_i$ and $W \subset V_j$ of size $|U| = |W| = \lceil \frac{n}{40k} \rceil$ be given. Let $N = \lceil \frac{n}{40k} \rceil^2$. Then the probability that there is no edge between U and W is $(1-p)^N \leq \exp(-pN) \leq \exp(-k^8n/1600)$. Hence the probability that the statement of (ii) does not hold can be bounded similarly as above by

$$k^2 2^{2n} \exp(-k^8n/1600) \;\leq\; k^2 \exp((2 - k^8/1600)n) \;\leq\; k^2 \exp(-2n) \;\leq\; 1/10.$$

Ad (iii). Let U and W be as in the statement of (iii) and let $q := |U| = |W|$. If there are at least $qk^{10}/2$ edges between U and W then there exists a set $E \subseteq U \times W$ of size $\lceil qk^{10}/2 \rceil$ such that all edges of E are contained in G. The probability of this event is at most

$$\binom{q^2}{\lceil qk^{10}/2 \rceil} p^{\lceil qk^{10}/2 \rceil} \quad \leq \quad \left(\frac{eq^2}{\lceil qk^{10}/2 \rceil} \cdot \frac{k^{10}}{n} \right)^{\lceil qk^{10}/2 \rceil} \quad \leq \quad \left(\frac{2eq}{n} \right)^{\lceil qk^{10}/2 \rceil}.$$

Summing over all possible choices for U and W we get that the probability that G does not fulfill the statement of (iii) is at most

$$\sum_{q=1}^{\lfloor n/40 \rfloor} k \binom{n}{q} \binom{kn}{q} \left(\frac{2eq}{n} \right)^{qk^{10}/2} \quad \leq \quad \sum_{q=1}^{\lfloor n/40 \rfloor} \left(\frac{ekn}{q} \right)^{2q} \left(\frac{q}{n} \right)^{qk^{10}/4} \left(\frac{4e^2 q}{n} \right)^{qk^{10}/4}$$

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$$\leq \sum_{q=1}^{\lfloor n/40 \rfloor} (ek)^{2q} \left(\frac{4e^2 q}{n} \right)^{qk^{10}/4},$$

where we used the fact that $k^{10} \geq 8$. Using furthermore that $(ek)^{8/k^{10}} \leq (e \cdot 3)^{8/3^{10}} \leq 9/e^2$ we can bound the last sum by

$$\sum_{q=1}^{\lfloor n/40 \rfloor} \left(\frac{(ek)^{8/k^{10}} 4e^2 q}{n} \right)^{qk^{10}/4} \leq \sum_{q=1}^{\lfloor n/40 \rfloor} \left(\frac{36q}{n} \right)^{2q}.$$

Because the function $f(q) = \left(\frac{36q}{n}\right)^{2q}$ is convex in q, $f(q)$ can be bounded by $\max\{f(1), f(\lfloor n/40 \rfloor)\}$ in the interval $[1, \ldots, \lfloor n/40 \rfloor]$. Therefore the sum is at most

$$\frac{n}{40} \max\left\{ \left(\frac{36}{n}\right)^2, \left(\frac{36}{40}\right)^{\frac{n}{20}} \right\} \leq \max\left\{ \frac{36}{n}, \frac{n}{40} \left(\frac{36}{40}\right)^{\frac{n}{20}} \right\}.$$

The function $\frac{n}{40}\left(\frac{36}{40}\right)^{\frac{n}{20}}$ is decreasing for $n \geq 3^{10}$ and we can set $n = 3^{10}$ in the maximum. Hence the maximum is at most $1/10$ which is what we were heading for.

Ad (iv). The expected number of cycles of length i is at most $(kn)_i p^i/(2i)$. Therefore the expected number of vertices contained in some cycle of length $3, \ldots, g-1$ is at most

$$\sum_{i=3}^{g-1} (kn)_i p^i \leq \sum_{i=3}^{g-1} k^i k^{10i} \leq \frac{k^{11g} - 1}{k^{11} - 1} \leq \frac{n-1}{k^{11} - 1} \leq \frac{n}{k^{10}}$$

by the definition of g. By Markov's inequality the probability that the number of vertices contained in a cycle of length $3, \ldots, g-1$ is more than $\frac{n}{4k} - 1$ can be bounded by $\frac{n}{k^{10}}/(\frac{n}{4k} - 1) \leq 4/k^8 \leq 1/10$.

Ad (v). Let $N = n^2 k(k-1)/2$ be the number of possible edges. If there are more than $\frac{n}{4k}$ vertices of degree more than $5k^{13}$ then the total number of edges is more than $\frac{n}{4k} \cdot 5k^{13}/2 = 5k^{12}n/8 \geq 5Np/4$. Let Y be the random variable counting the number of edges. Then Y has binomially distribution $\mathcal{B}(N, p)$. Therefore, by Chebychev's inequality,

$$Prob[Y \geq \frac{5}{4} Np] \leq \frac{16 Var(Y)}{(\mathbf{E}(Y))^2} = \frac{16Np(1-p)}{(Np)^2} \leq \frac{16}{Np} \leq \frac{32}{k^{12} n} \leq \frac{1}{10}.$$

□

Let $k \geq 2$ and $n \geq k$ be integers. We now show how to turn a graph $G \in \mathcal{F}(n,k)$ into a uniquely k-colourable graph with girth at least $g := \lfloor \frac{1}{11} \frac{\log n}{\log k} \rfloor$. To do this we first remove the vertices contained in some short cycle and the vertices whose degree is too large. Then we remove the vertices which have too few neighbours in another partition class. Precisely we do the following.

Procedure Alter
Input: A graph $G \in \mathcal{F}(n,k)$, $n \geq k \geq 2$.
Output: A graph $H \in \mathcal{G}(m,k)$ for an $m \leq n$.

1. Let $C := \{v : v \text{ is a vertex contained in a cycle of length } 3, \ldots, g-1\}$.
2. Let $Y := \{v : v \text{ has degree larger than } 5k^{13} \text{ in } G\}$.
3. Let $\ell := \max\{|V_i \cap (C \cup Y)| : i \in [k]\}$.

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4. Remove from every V_i exactly ℓ vertices in such a way that all vertices from $C \cup Y$ are removed. Call the remaining sets W_i.
5. WHILE there exist $i, j \in [k]$ and a $v \in W_i$ such that v has less than $k^{10}/2$ neighbours in W_j DO
   remove v from W_i and remove for every $r \neq i$ an arbitrary vertex from W_r.
6. Output the graph $H = G[W_1 \cup \ldots \cup W_k]$.

Observe that the sets W_i all have the same size.

Lemma 2.2. *The WHILE-loop is executed at most $\frac{n}{2k}$ times.*

Proof. Assume that the WHILE-loop is executed more than $\frac{n}{2k}$ times. Let W_i', $i \in [k]$, be the sets W_i after $\lceil \frac{n}{2k} \rceil$ executions of the WHILE-loop. Then $|W_i'| \geq (k-1)n/k$ as $\ell \leq \frac{n}{2k} - 1$ because G has properties (iv) and (v).
There exist $i, j \in [k]$ such that the tuple (i, j) was selected more than $\lceil \frac{n}{2k^3} \rceil$ times in the WHILE-loop. I.e. there exists a set $U \subseteq V_i$ of size $\lceil \frac{n}{2k^3} \rceil$ such that all vertices from U have less than $k^{10}/2$ neighbours in W_j'. Let W be a subset of W_j' of size $\lceil (k-1)n/k \rceil$. Then there are less than $k^7 n/4$ edges between U and W and contradicting property (i). □

Lemma 2.3. *Let a graph H be an output of the procedure* Alter. *Then H has the following properties.*
(a) H has a k-colouring all of whose colour classes are of equal size m, where $(k-1)n/k \leq m \leq n$.
(b) H has girth at least $\lfloor \frac{1}{11} \frac{\log n}{\log k} \rfloor$ and maximal degree at most $5k^{13}$.
(c) H is uniquely k-colourable.

Proof. (a) and (b) follow immediately from the construction and from Lemma 2.2.
To prove (c), as in the procedure, denote by W_i for $i \in [k]$ the colour classes of the colouring arising from the partition. Assume that there is another k-colouring of H. Then this colouring has a colour class of size at least m which does not coincide with any of the W_i. Hence we may fix an independent set A in H of size m which does not coincide with any of the W_i. We distinguish two cases.
Consider first the case that there is an $i \in [k]$ such that

$$|W_i \backslash A| \leq \frac{n}{40}. \tag{2}$$

Let $U := W_i \backslash A$ and $|U| \leq n/40$. Let furthermore $W := A \setminus W_i$. Then $|W| = |U|$. As A is an independent set all neighbours of vertices from W in W_i must belong to U. Therefore, by step (5) of the algorithm Alter above, there are at least $|W| k^{10}/2$ edges between U and W which contradicts property (iii).
Assume now that for every $i \in [k]$, (2) does not hold. Select an index i such that $|A \cap W_i| \geq m/k$ and note that $m/k \geq \lceil \frac{n}{40k} \rceil$ since $k \geq 2$. As $|A \setminus W_i| = |W_i \backslash A| > n/40$, there also is an index $j \neq i$ such that $|A \cap W_j| \geq \lceil \frac{n}{40k} \rceil$. As A is an independent set, this contradicts property (ii). □

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Theorem 1.1 now follows easily from the previous lemmas.

Proof of Theorem 1.1. If $k = 2$ then let G be the path on $N = N_1$ vertices.
If $k \geq 3$ and $N_1 \leq k^{33}$ then let G be any complete k-partite graph on $N = N_1$ vertices which of course has girth $3 \geq \frac{\log N}{11 \log k}$.
Let now $k \geq 3$ and $N_1 \geq k^{33}$ be given. Define $n := \lfloor N_1/k \rfloor$. Then $n \geq k^{10}$ and by Lemma 2.1 $\mathcal{F}(n,k) \neq \emptyset$. Apply then ALTER to a graph $G \in \mathcal{F}(n,k)$. Let H be the output of the procedure. Then H has $N = km$ vertices by Lemma 2.3, where $N_1 \geq N = km \geq (k-1)\lfloor N_1/k \rfloor \geq (1-1/k)N_1 - k \geq 2N_1/3 - k \geq N_1/2$. Furthermore H has girth at least

$$\left\lfloor \frac{1}{11} \frac{\log n}{\log k} \right\rfloor \quad \geq \quad \left\lfloor \frac{1}{11} \frac{\log N_1 - \log k}{\log k} \right\rfloor \quad \geq \quad \left\lfloor \frac{1}{12} \frac{\log N_1}{\log k} \right\rfloor$$

and maximal degree at most $5k^{13}$.

To show the existence of a randomized algorithm as claimed in the theorem observe that if ALTER is applied to a graph $G \in \mathcal{G}(n,k) \setminus \mathcal{F}(n,k)$ it still outputs a k-colourable graph with required girth and maximal degree. □

3. Hardness Results

Proof of Theorem 1.2. Given a polynomial algorithm $\mathcal{A}$ for the decision problem k-COLOURABILITY on the class of graphs G of girth $g(G) \geq \lfloor \frac{\log |G|}{13 \log k} \rfloor$ and maximum degree $\Delta(G) \leq 6k^{13}$ we exhibit a co-RP algorithm for k-COLOURABILITY. It then follows that NP $\subseteq$ co-RP and hence NP = ZPP, so in particular we have NP=RP.

Let G be a graph. We may assume that $|G| \geq k^{5/2}$. Apply Theorem 1.1 to $N_1 := |G|^{24}$ and k to construct in randomized polynomial time a k-colourable graph H with $|H| \leq N_1$ that has girth $g(H) \geq \lfloor \frac{\log N_1}{12 \log k} \rfloor \geq 5$ and maximal degree $\Delta(H) \leq 5k^{13}$ together with a k-coloring of H.

Consider a shortest cycle C in H. Choose a vertex x on C. Consider the two resp. three vertices at distance at least $\lceil |C|/2 \rceil - 1$ from x on C. Then one of these vertices, say y, is coloured different from x and fulfills $dist_H(x,y) \geq \lceil g(H)/2 \rceil - 1$. Now replace each edge $\{u,v\}$ in G by a copy of H, identifying u with x and v with y to obtain a graph G_1. Since every cycle in G_1 that is not completely contained in a copy of H has length at least $3 \cdot dist_H(x,y) \geq 3\lceil g(H)/2 \rceil - 3 \geq g(H)$, the graph G_1 has girth $g(G_1) \geq g(H)$.

The maximal degree of G_1 is at most $|G| \cdot 5k^{13}$. One of the colour classes, say S, in the k-colouring of H has size at least $|H|/k \geq N_1/(2k) \geq 5|G|$. Replace each vertex v in G_1 of degree larger than $6k^{13}$ by a new copy of H connecting each neighbour of v in G_1 to a vertex from S in such a way that no vertex from S gets more than k^{13} new neighbours. We obtain a graph G_2 on at most $(\binom{|G|}{2} + |G|)N_1 \leq |G|^2 N_1$ vertices with maximal degree at most $6k^{13}$. Moreover, G_2 has girth

$$g(G_2) \geq g(G_1) \geq g(H) \geq \left\lfloor \frac{\log N_1}{12 \log k} \right\rfloor = \left\lfloor \frac{\log N_1 + 2 \log |G|}{13 \log k} \right\rfloor \geq \left\lfloor \frac{\log |G_2|}{13 \log k} \right\rfloor .$$

The co-RP algorithm for k-COLOURABILITY now outputs $\mathcal{A}(G_2)$. It is then clear that the algorithm runs in randomized time polynomially bounded in $|G|$. Furthermore, if G is

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k-colourable so is G_2 by the choice of x and y. On the other hand, if G is not k-colourable then G_2, too, is not k-colourable provided H is uniquely k-colourable which happens with probability $1/2$. □

Proof of Corollary 1.2. (a) For fixed k and g a uniquely k-colourable graph on at most $k^{12(g+1)}$ vertices with girth at least g and maximum degree at most $5k^{13}$ (guaranteed to exist by Corollary 1.1) can be constructed in constant time by complete enumeration. The reduction is then similar to the one above.
(b) Let $g := 1+\max\{g(G) : G \in \mathcal{G} \text{ and } g(G) < \infty\}$. Then the class of all graphs of girth at least g and of maximum degree at most $6k^{13}$ is contained in $\mathcal{F}orb\ (\mathcal{G})$. Therefore, (b) follows immediately from (a). □

Proof of Theorem 1.4. We reduce from k-COLOURABILITY. Let $G = (V, E)$ be a graph.
Suppose G has a vertex $v \in V$ of degree $d(v) > k + \lceil\sqrt{k}\rceil - 1$. Add vertices $r_1, \ldots, r_{\lceil\sqrt{k}\rceil}$ to G, detach k of the edges incident with v and attach them evenly to $r_1, \ldots, r_{\lceil\sqrt{k}\rceil}$ instead. Finally, add a $(k-1)$-clique C to the graph and connect v and each r_i, $1 \le i \le \lceil\sqrt{k}\rceil$, to each $c \in C$ by an edge. Obviously, the graph G' arising this way is k-colorable if and only if G is. The newly introduced vertices all have degree at most $k + \lceil\sqrt{k}\rceil - 1$ and the degree of v has decreased by one. Therefore, after $\mathcal{O}(m(G))$ such operations the graph produced has maximum degree at most $\Delta(G) \le k + \lceil\sqrt{k}\rceil - 1$ as required. □

We conclude this section with a study of k-COLOURABILITY for an integer-valued, non-decreasing function $k = k(|G|)$.

Theorem 3.1. *Let $\varepsilon > 0$ be a constant. Let k be an integer valued non-decreasing function such that for every integer $n \ge 3$,*

$$3 \quad \le \quad k(n) \quad \le \quad n - n^{\varepsilon} + 3 \tag{3}$$

holds. Then the problem k-COLOURABILITY is NP-complete.

Proof. We reduce from 3-COLOURABILITY. Let G be an instance of 3-COLOURABILITY, $n := |G|$. Let k be an integer valued function as in the statement of the theorem. Define for every non-negative integer N the function f by $f(N) := k(N+n) - N - 3$. Then by (3) $f(0) \ge 0$. Let $\ell := \lceil 1/\varepsilon \rceil$. By the right hand side of (3)

$$f(n^{\ell} - n) = k(n^{\ell}) - n^{\ell} + n - 3 \le -n^{\ell\varepsilon} + n \le 0.$$

Because k is non-decreasing, $f(N+1) \ge f(N) - 1$. Hence there is a minimal integer N_0 with $f(N_0) = 0$. Then $0 \le N_0 \le n^{\ell} - n$. Now define a graph H by adding an N_0-clique to G connecting every vertex from the clique with every vertex from G. This graph has at most n^{ℓ} vertices and therefore the reduction is polynomial. Furthermore, H is $k(|H|)$-colourable if and only if G is 3-colourable. □

On the other hand, let c be a fixed integer and define $k(n) := n - c$. Then k-COLOURABILITY is in P because a graph is k-colourable if and only if it contains a c-colourable subgraph of order $2c$.

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Theorem 3.1 implies the following strengthening of Theorem 1.2.

Corollary 3.1. *For some $\varepsilon > 0$, let $3 \le k(n) \le n - n^{\varepsilon} + 3$ be an integer-valued non-decreasing function. Then unless* RP=NP, *there exists no polynomial algorithm for the problem k-*COLOURABILITY *on the class of graphs G with girth $g(G) \ge \lfloor \frac{\log|G|}{13 \log k(|G|)} \rfloor - 1$.*

Proof. If $k \ge |G|^{1/39}$ then $\frac{\log|G|}{13 \log k} \le 3$ and the statement follows directly from Theorem 3.1. For $k \le n^{1/39}$, we reduce from ℓ-COLOURABILITY, where ℓ is the function $\ell(n) := k(n^{26})$. Then ℓ fulfills condition (3) of Theorem 3.1 for an appropriately chosen $\varepsilon > 0$. The rest of the proof is then similar as the proof of Theorem 1.2 apart that we only have $g(H) \ge 3$ and thus only $g(G_1) \ge g(H) - 1$. □

4. Discussion

There is couple of natural complexity theoretic questions related to unique colourability. The problem "given a graph G, is G uniquely k-colourable?" for example is NP-hard [2], the problem "given a graph G and a k-colouring of G, is there another k-colouring of G?" is NP-complete [6]. If there exists a polynomial-time algorithm which finds a 3-colouring of a uniquely 3-colourable graph then NP=RP [24]. For complexity classes related to uniqueness problems and a survey of related results, see [15, 25].

We close by stating some open questions:

- What is the smallest $f(k)$ such that k-COLOURABILITY on graphs of maximum degree $\Delta \le f(k)$ is still NP-complete ?
- Or, more general, for given $k \ge 3$, give a complete characterization of all classes of graphs $\mathcal{G}$ having the property that k-COLOURABILITY is NP-complete on $\mathcal{F}orb\ (\mathcal{G})$ (cf. Corollary 1.2 (b)).
- For which classes of graphs $\mathcal{G}$ are there k-colourable graphs in $\mathcal{F}orb\ (\mathcal{G})$?
- Can one improve k to $k = n - \log n$ in Theorem 3.1?

The same questions can also be formulated for induced forbidden subgraphs. The third question for $k = 3$ and the induced case was recently investigated by RANDERATH (personal communication).

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