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Surface Realization with the Intersection Edge Functional

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Abstract

Deciding realizability of a given polyhedral map on a (compact, connected) surface belongs to the hard problems in discrete geometry, from the theoretical, the algorithmic, and the practical point of view.

In this paper, we present a heuristic algorithm for the realization of simplicial maps, based on the intersection edge functional. The heuristic was used to find geometric realizations in $\mathbb{R}^3$ for *all* vertex-minimal triangulations of the orientable surfaces of genus $g = 3$ and $g = 4$. Moreover, for the first time, examples of simplicial polyhedra in $\mathbb{R}^3$ of genus 5 with 12 vertices were obtained.

1 Introduction

A *polyhedral map* on a surface is a (finite) set of polygons (with at least three sides), which are glued together (topologically) along edges to form the surface, such that there are no self-identifications on the boundaries of the polygons, and two polygons are either disjoint or intersect in exactly one edge or one vertex only. We thus can think of a polyhedral map as a combinatorial model for a surface.

For a given polyhedral map it is natural to try to visualize it as a *polyhedron* in three-space (or in higherdimensional space $\mathbb{R}^d$) such that every polygon is the convex hull of its vertices and two polygons are either disjoint in $\mathbb{R}^d$, they intersect in a common edge and are not coplanar, or they intersect in a common vertex only. Such a realization usually is called a *geometric* or *polyhedral realization*, with straight edges, plane polygons, and no non-trivial intersections (with neighboring polygons being not coplanar).

Example 1: A polyhedral map on the 2-sphere S^2 consisting of the polygons 123, 12478, 13568, 2354, 4567, and 678 together with a corresponding realization in $\mathbb{R}^3$ is displayed in Figure 1.

Realizability of maps on the 2-sphere S^2 was proved by Steinitz ([48], [49]; cf. also [27, Ch. 13], [60, Lec. 4]): Every polyhedral map on the 2-sphere S^2 is geometrically realizable in $\mathbb{R}^3$ as the boundary complex of a convex 3-polytope.

However, not all polyhedral maps are realizable. For example, *simple polyhedral maps* (i.e., maps with all vertices of valence three) on surfaces different from the 2-sphere S^2 are not realizable in any $\mathbb{R}^d$ (see Grünbaum [27, Ex. 11.1.7, Ex. 13.2.3]).

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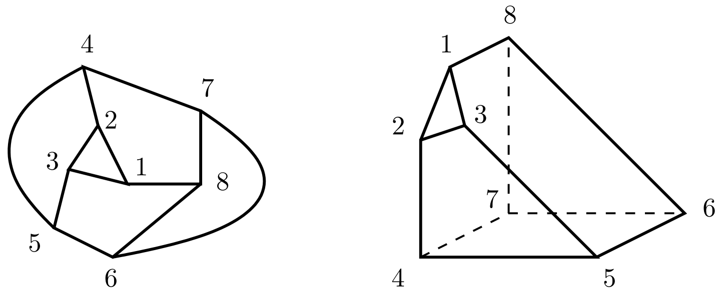

Figure 1: A polyhedral map on S^2 and a corresponding geometric realization in $\mathbb{R}^3$.

Example 2: All 6-3-*equivelar* maps on the torus (i.e., maps consisting of only 6-gons with every vertex belonging to exactly three 6-gons) are simple and therefore cannot be realized in any $\mathbb{R}^d$. The smallest example (see Figure 2) of the family is the combinatorial dual of Möbius' 7-vertex triangulation of the torus [39]. A "realization" of this torus with flat, but non-convex 6-gons was given by Szilassi [55], the *Szilassi-torus.*

Betke and Gritzmann [8] further found the following combinatorial obstruction to geometric realizability: Let W be any subset of the set of odd valent vertices of a polyhedral map M^2 and let F_W be the set of facets containing some vertex of W. If $2|F_W| \leq |W|$, then M^2 is not realizable in any $\mathbb{R}^d$.

Again, the Betke-Gritzmann obstruction rules out realizability of 6-3-equivelar maps on the torus. The obstruction was also used by McMullen, Schulz, and Wills [38] to show non-realizability for other, non-simple families of equivelar maps.

Apart from Steinitz' theorem and the two above obstructions, rather little is known on the realizability of (polyhedral maps on) orientable surfaces in $\mathbb{R}^3$.

In a *simplicial (i.e., triangulated) map* every triangle contains at most three odd valent vertices, from which it can be deduced that $|F_W| \geq |W|$ for every subset W of odd valent vertices. In particular, the Betke-Gritzmann obstruction cannot be applied to simplicial maps to show non-realizability. Almost 40 years ago, Grünbaum proposed:

Conjecture 1 (Grünbaum [27]) *Every triangulated torus is realizable geometrically in* $\mathbb{R}^3$*.*

Until recently, no computational tools were available *to actually find realizations* for orientable simplicial surfaces, not even for examples of small genus with few vertices: In the past 25 years the most promising approach to obtain a polyhedral realization for a given triangulation was to try to *build a physical model*, for example, with the *rubber band technique* of Bokowski [9].

In this paper, we present a heuristic algorithm for finding polyhedral realizations for orientable simplicial surfaces, which, for the first time, gives stronger results than the physical approach. In particular, we will show that all vertex-minimal triangulations of orientable surfaces of genus $g = 3$ and $g = 4$ are realizable and that there are examples of vertex-minimal simplicial polyhedra of genus 5 with 12 vertices.

In the following Section we give a brief survey on realizability results for surfaces, minimal triangulations, and algorithmic aspects of deciding realizability. Our realization

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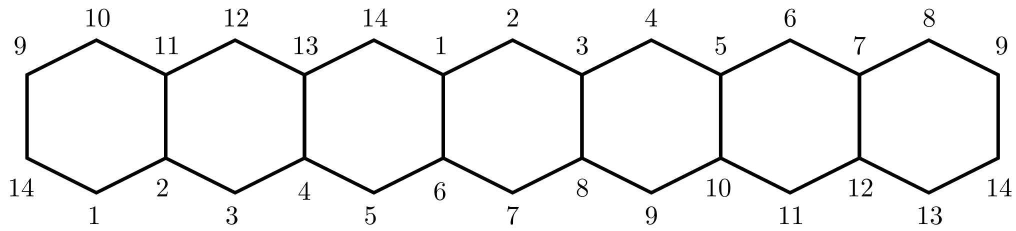

Figure 2: The non-realizable 6-3-equivelar map with 14 vertices on the torus.

heuristic is then presented in Section 3 and computational results are given in Section 4. An extension of our approach to convex realizations of triangulated spheres is discussed in Section 5.

2 Realizability of Polyhedral Surfaces and Polyhedral Complexes

In general, every d-dimensional simplicial complex (with n vertices) is polyhedrally embeddable in $\mathbb{R}^{2d+1}$, as it can be realized as a subcomplex of the boundary complex of the cyclic polytope $C(n, 2d+2)$; cf. Grünbaum [27, Ex. 25, p. 67]. However, van Kampen [57] and Flores [26] showed that d-dimensional simplicial complexes cannot always be embedded topologically in $\mathbb{R}^{2d}$, e.g., the d-skeleton $\mathrm{Sk}_d(\Delta_{2d+2})$ of the $(2d+2)$-simplex Δ_{2d+2} is not embeddable in $\mathbb{R}^{2d}$ (for further examples and references see Matoušek [37, 5.1], Novik [40], and Schild [46]).

For smooth d-manifolds, Whitney [58] proved that they can smoothly be embedded in $\mathbb{R}^{2d}$, and Penrose, Whitehead, and Zeeman [41] showed that for $0 < 2(k+1) \leq d$ every k-connected PL (i.e., piecewise linear) d-manifold has a PL embedding in $\mathbb{R}^{2d-k}$. In particular, surfaces have PL embeddings in $\mathbb{R}^4$. Orientable surfaces (with or without boundary) and non-orientable surfaces with boundary are even PL embeddable in $\mathbb{R}^3$ (which follows from the classification of surfaces by Dehn and Heegaard [24]). Closed non-orientable d-manifolds cannot be embedded topologically in $\mathbb{R}^{d+1}$; cf. Bredon [15, p. 353].

Thus, for triangulated orientable surfaces (with or without boundary) and for triangulated non-orientable surfaces with boundary we have:

- PL embeddability in $\mathbb{R}^3$
- and polyhedral realizability in $\mathbb{R}^5$.

Triangulations of closed non-orientable surfaces are

- not (topologically) embeddable in $\mathbb{R}^3$,
- but are PL embeddable in $\mathbb{R}^4$,
- and are polyhedrally realizable in $\mathbb{R}^5$.

Perles showed (cf. [27, 11.1.8]) that a polyhedral map is realizable in some $\mathbb{R}^d$ if and only if it is realizable in $\mathbb{R}^5$.

A natural approach to establish geometric realizability in $\mathbb{R}^3$ for orientable surfaces of genus $g > 1$ is to identify a given polyhedral map as a subcomplex of the boundary complex of a convex 4-polytope P. The Schlegel diagram of P then yields coordinates for the realization in $\mathbb{R}^3$; see, for example, McMullen, Schulz, and Wills [38] for realizations of equivelar maps obtained this way, and cf. Altshuler [1], [2] for combinatorial properties on maps that guarantee realizability via Schlegel diagrams.

Altshuler and Brehm [4] gave a polyhedral map T_8 on the torus with only 8 vertices which is realizable (cf. also Simutis [47]), but *not* via Schlegel diagrams. In fact, the map T_8 is not isomorphic to a subcomplex of the boundary complex of any convex polytope [4].

Realizability (via subcomplexes of convex 5-polytopes) of triangulations of the torus and the projective plane in $\mathbb{R}^4$ was proved by Brehm and Schild [19], herewith sharpening Barnette's result [6] on the geometric realizability of triangulations of the projective plane in $\mathbb{R}^4$.

Polyhedral surfaces that are obtained by projections (of 2-dimensional subcomplexes) of higherdimensional polytopes together with obstructions to projectability are discussed by Sanyal, Schröder, and Ziegler [44].

For further results and references on polyhedral maps see Brehm and Wills [21], Brehm and Schulte [20], and Ziegler [59].

2.1 Simplicial Maps

Let M be a (closed) triangulated surface with $n = f_0$ vertices, f_1 edges, and f_2 triangles, i.e., M has *face-vector* $f = (n, f_1, f_2)$. If M has Euler characteristic $\chi(M)$, then by Euler's equation,

$$n - f_1 + f_2 = \chi(M).$$

Double counting of the incidences between edges and triangles of a triangulation yields $2f_1 = 3f_2$. So together,

$$f = (n, 3n - 3\chi(M), 2n - 2\chi(M)).$$

According to Heawood's bound [29] from 1890, triangulations of a 2-manifold M have at least

$$n \geq \left\lceil \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \right\rceil \tag{1}$$

vertices. Heawood's bound is sharp for all surfaces, except for the cases of the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3, where an extra vertex has to be added to the lower bound, respectively.

Corresponding vertex-minimal triangulations of the real projective plane $\mathbb{R}\mathbf{P}^2$ with 6 vertices and of the 2-torus with 7 vertices (Möbius' torus [39]) were already known in the 19th century, but it took until 1955 to complete the construction of series of examples of minimal triangulations for all non-orientable surfaces (Ringel [43]) and until 1980 for all orientable surfaces (Jungerman and Ringel [32]).

If a given triangulation of an orientable surface is realizable in $\mathbb{R}^3$, then so are subdivisions of it. Thus, vertex-minimal triangulations apparently are good candidates for non-realizable maps. Hereby, triangulations with

$$n = \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \tag{2}$$

are of particular interest (cf. [23]), as for these we have $f_1 = \binom{n}{2}$, that is, the respective triangulations are *neighborly* with complete 1-skeleton (which should make realizability difficult).

A polyhedral realization of the combinatorially unique vertex-minimal 7-vertex triangulation of the torus with $f = (7, 21, 14)$ was given by Császár [23], [34] (although realizability was possibly known already to Möbius; cf. [39, p. 553], [42]).

The next case of equality (2) yields 59 examples of vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 [3]; see below.

2.2 Realizability vs. Non-Realizability of Simplicial Maps

For every individual triangulation of an orientable surface, realizability (in $\mathbb{R}^3$) can be decided algorithmically by the following two-step procedure, cf. [10], [14, Ch. VIII]:

1. Enumerate all oriented matroids compatible with the given triangulation. If there are none, then the triangulation is not realizable, else
2. decide realizability of the oriented matroids from 1. via solving associated polynomial inequality systems.

Theoretically, the second step can be done algorithmically (for example, with Collin's *Cylindrical Algebraic Decomposition* algorithm [22]). In practice, however, there are no methods known that would work sufficiently fast to yield results even for small examples. (For more comments on this and also for algebraic tools such as final polynomials see [10], [14, Ch. VIII].)

In a first breakthrough, Bokowski and Guedes de Oliveira [13] showed (using 10 CPU years) that one of the 59 vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 has no compatible orientable matroid and therefore is not realizable.

Recently, Schewe [45] substantially improved the enumeration of compatible orientable matroids and was able to show that, in fact, all 59 vertex-minimal 12-vertex triangulations of the orientable surface of genus 6 are non-realizable. Moreover, he found three examples of non-realizable vertex-minimal 12-vertex triangulations of the orientable surface of genus 5. At least for one of these examples it is possible to remove a triangle from the triangulation while maintaining non-realizability. Connected sums with other triangulations then still are non-realizable. Thus, for every orientable surface of genus $g \geq 5$ there are triangulations that cannot be realized geometrically in $\mathbb{R}^3$.

Apart from the approach via oriented matroids, non-realizability results (for simplicial maps in $\mathbb{R}^3$) seem to be difficult to achieve: Novik [40] associated an integer program with a given triangulation, which, if it has no solution, yields non-realizability. Improved systems have been proposed by Timmreck [56]. So far, however, for orientable surfaces all tested systems were found to have solutions or turned out to be computationally intractable. In a different approach, Brehm [17] used a linking number argument to show that there is a non-realizable triangulation of the Möbius strip with 9 vertices.

2.3 Heuristics for the Realization of Simplicial Maps

Until recently, it was considered to be rather difficult and time-consuming to actually find realizations for given triangulations. Examples of polyhedral realizations of vertex-minimal triangulations of the orientable surfaces of genus 3 and 4 with 10 and 11 vertices, respectively, were constructed *by hand* by Bokowski and Brehm [11], [12] and Brehm [16], [18]. Some of these examples were found by exploiting combinatorial symmetries of the triangulations, others with the *rubber band technique* of Bokowski [9].

A first (and simple) computer heuristic (*by choosing coordinates randomly*) was used in [36] to show that at least 864 of the 865 examples of vertex-minimal triangulations of the

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orientable surface of genus 2 are realizable. The remaining case then was settled by Bokowski with the rubber band method [9]. All 865 examples were later found to have realizations with *small coordinates* [30], i.e., all these examples are realizable with integer coordinates in general position in the $(4 \times 4 \times 4)$-cube. Moreover, realizations in the $(5 \times 5 \times 5)$-cube were obtained for 17 of the 20 vertex-minimal triangulations with 10 vertices of the orientable surface of genus 3 *by isomorphism-free enumeration* of possible coordinate configurations in general position [31].

In the following, we will discuss an improved heuristic to obtain polyhedral realizations in $\mathbb{R}^3$ for triangulations of orientable surfaces. In particular, we will show that all vertex-minimal triangulations of orientable surfaces of genus $g = 3$ and $g = 4$ are realizable and that there are examples of simplicial polyhedra of genus 5 with 12 vertices.

3 Realization with the Intersection Edge Functional

As mentioned in the previous section there have been so far three major heuristics for the realization of simplicial surfaces (of genus $g \geq 1$) in $\mathbb{R}^3$:

- by explicit geometric construction [11], [12], [16], [18] (e.g., via the rubber band technique of Bokowski [9]);
- by choosing coordinates randomly [36];
- by enumeration of realizations with small coordinates [30], [31].

As a more sophisticated approach we suggest to proceed as follows. For a given triangulation (at least for orientable surfaces of small genus)

1. start with random coordinates for the vertices of the triangulation
2. and then "move vertices around" to eventually obtain a realization.

For the second step we take as an objective to minimize the *intersection edge functional*:

> *Let M^2 be a triangulated orientable surface with vertex-set V and let $V_{\mathbb{R}^3}$ be a set of $|V|$ vertices in general position in $\mathbb{R}^3$. Then every pair of triangles of M^2 coordinatized with the coordinates of $V_{\mathbb{R}^3}$ either has empty intersection in $\mathbb{R}^3$ or intersects in an edge; see Figure 3 for the intersection edge u–x of two triangles. The sum of the lengths of the intersection edges over all pairs of (non-neighboring) triangles is the* intersection edge functional.

We require that the points are *in general position*, i.e., no three points are on a line and no four points are on a plane, in order to avoid degenerate intersections of triangles. Further, we use integer coordinates and therefore move the points in the second step above on the integer grid only.

Our aim will be to find integer coordinates in general position for which the intersection edge functional vanishes for the given triangulation.

From an initial set of random coordinates we proceed to minimize the intersection edge functional by a local search of *hill-climbing* type:

> *In every step, we randomly pick a vertex $v \in V_{\mathbb{R}^3}$ and a coordinate direction, $\pm x$, $\pm y$, or $\pm z$, and then move the vertex v one integer step into the respective direction. If the resulting set of coordinates is in general position and the new value of the intersection edge functional is strictly smaller than before, the move*

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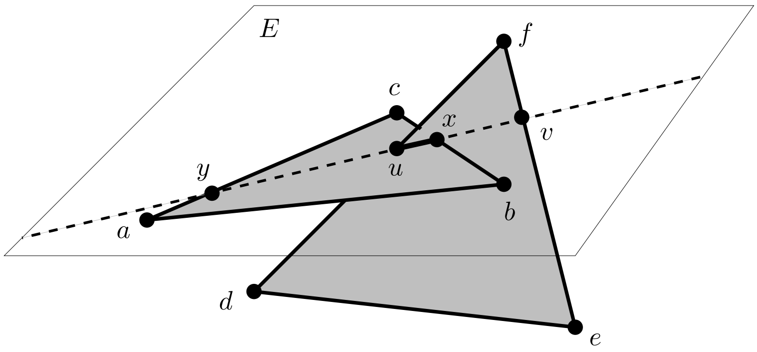

Figure 3: Two intersecting triangles.

is accepted and the next step is executed. Otherwise the move is discarded and we start anew from the previous set of coordinates.

If all possible choices of moves have been tested for some set of coordinates without improvement, then we are stuck in a local minimum. *In this case, for one step only, we choose one of the* admissible moves*, i.e., a move that yields a set of coordinates in general position, but which not necessarily decreases the intersection edge functional. From there, we then try to continue to decrease the intersection edge functional in a new direction.*

Example 3: Local minima can occur even for small triangulations. For example, the boundary of the octahedron with triangles

123 124 135 145 236 246 356 456

and furnished with coordinates

1: (4,4,6) 2: (5,6,6) 3: (9,7,4) 4: (5,9,1) 5: (4,6,3) 6: (1,5,7)

attains a local minimum for the intersection edge functional with value 3.17. Figure 4 provides a visualization.

3.1 Details of the Algorithm

Initially the vertices of the triangulation are placed randomly at general positions in a $(50 \times 50 \times 50)$-cube. This cube is chosen at the center of a larger $(250 \times 250 \times 250)$-cube that we take as *bounding box* for all possible positions of the vertices during the local search. After the choice of the starting positions the smaller cube is not used anymore.

- Thus, we allow the diameter of the vertex-set to increase moderately (which possibly helps to decrease the intersection edge functional by unfolding the initial shape).
- At the same time there is a fixed lower bound for the change, at every step, of the intersection edge functional (determined by the size of the bounding box and the fact that we admit integer coordinates only). This way we avoid that the sequence of improvements for the functional converges to zero.

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Figure 4: A non-realization of the octahedron with locally minimal functional.

An *admissible step* then is a movement of one vertex by one integer in one of the coordinate directions such that the resulting set of coordinates is in general position and is within the bounding box.

- If the intersection edge functional becomes zero, a realization for the given triangulation is found.
- If a realization is not found within a fixed period of time T, the whole process is restarted for the triangulation, beginning with the random selection of the starting coordinates (in the smaller cube). In doing so we try to overcome situations in which the process cycles between different local minima.

A standard problem with local search algorithms is to appropriately choose the parameters that govern the procedure. For some of the 20 examples of vertex-minimal 10-vertex triangulations of the orientable surface of genus 3 we tried the following variants:

- We chose different sizes for the initial cube, ranging from $5 \times 5 \times 5$ to $500 \times 500 \times 500$.
- We allowed the bounding box to be between 1 up to 8 times the size of the initial cube.
- If the edge functional decreases by moving one vertex in one direction, we moved the vertex as far as possible in this direction (until the intersection edge functional starts to increase again).
- In case of a local minimum we determined all pairs of vertices for which the exchange of their positions decreases the intersection edge functional. We then executed one such exchange at random. If there is no such pair, we randomly exchanged two arbitrary vertices.
- Instead of minimizing the intersection edge functional we tried to minimize the *normalized intersection edge functional*, which is obtained from the intersection edge functional by dividing by the total length of the edges of the polyhedral map.
- We first generated 10000 sets of initial coordinates of which we selected the set with the smallest functional before starting the local search.

From all these variants the previously described one turned out to have the best performance. This variant then was used to find realizations for other triangulations.

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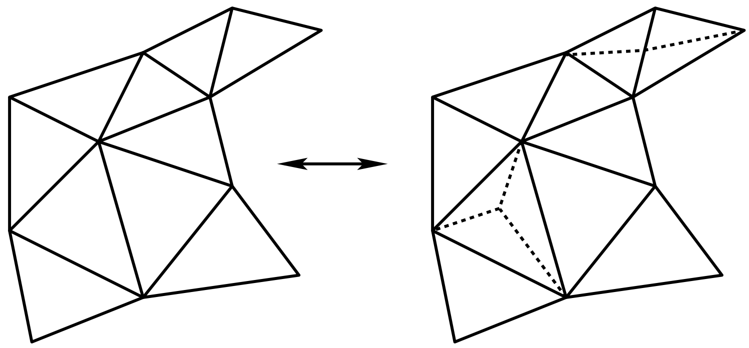

Figure 5: Subdivisions of a triangle and of an edge.

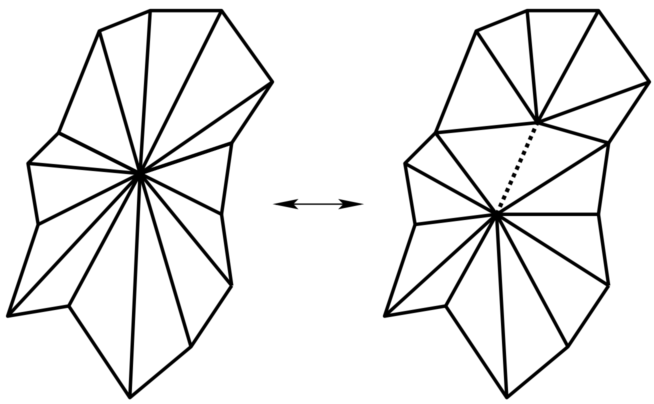

Figure 6: Expansion (respectively contraction) of an edge.

3.2 Examples of Minimal Triangulations

If some triangulation of an orientable surface is realizable, then so are all subdivisions of it: In the case of the two elementary subdivisions of a triangle and of an edge of the triangulation (see Figure 5), we can always place the new vertex slightly above or below the respective triangle or the respective edge of the given realization. For finer subdivisions, we can proceed iteratively.

For orientable surfaces of genus $g \geq 1$, the set of minimal triangulations that are not subdivisions is infinite, as it comprises all triangulations that have only vertices of degree at least 5. A finite subset of particular interest to test realizability is the set of vertex-minimal triangulations. If these are realizable, this should give a strong indication that, in fact, all triangulations of the surface are realizable.

A somewhat larger, but still finite set of triangulations that are not subdivisions is defined as follows: If we take *edge expansions* (with *edge contractions* as inverses; see Figure 6) instead of subdivisions, then for every surface there is only a finite set (see Barnette and Edelson [7]) of *irreducible* triangulations for which no edge can be contracted without changing the topological type of the triangulation. However, it is not clear whether a realizable

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Table 1: Numbers of vertex-minimal triangulations of the orientable surfaces of genus $g \leq 6$.

g	$n_{\min}$	Types
0	4	1
1	7	1
2	10	865
3	10	20
4	11	821
5	12	751.593
6	12	59

triangulation of an orientable surface of genus $g \geq 1$ remains realizable after the expansion of an edge.

It follows from the work of Steinitz [49, §46] that every triangulated 2-sphere can be reduced to the boundary of the tetrahedron by a sequence of edge contractions. In other words, the boundary of the tetrahedron is the only irreducible triangulation of the 2-sphere.

Grünbaum and Lavrenchenko [33] determined the number of irreducible triangulations of the torus: there are 21 such examples with up to 10 vertices and they are all realizable. Recently, Sulanke [51], [52], [53] has shown that there are exactly 396.784 examples of irreducible triangulations (with up to 17 vertices) of the orientable surface of genus 2.

Although it might be desirable to test realizability for a larger set of irreducible triangulations, we restricted ourselves to vertex-minimal ones. There is only one unique vertex-minimal triangulation of the torus, i.e., Möbius 7-vertex torus [39] for which Császár [23] gave an explicit polyhedral model. Vertex-minimal triangulations of the orientable surfaces of genus 2 and 3 were enumerated in [36], those of genus 4 and 5 in [54], and the vertex-minimal examples of genus 6 in [3]; see Table 1 for the corresponding minimal numbers of vertices $n_{\min}$ and the respective numbers of combinatorial types of triangulations.

4 Computational Results

Geometric realizations for the 865 vertex-minimal triangulations of the orientable surface of genus 2 were found in [9] and [36]; see also [30] for corresponding realizations with small coordinates. Moreover, for 17 of the 20 vertex-minimal triangulations of the orientable surface of genus 3 realizations with small coordinates were obtained in [31]. Thus, the first task for our program was to realize the remaining three examples.

Theorem 2 *All* 20 *vertex-minimal* 10*-vertex triangulations of the orientable surface of genus* 3 *are geometrically realizable in* $\mathbb{R}^3$.

The resulting sets of coordinates for the realizations can be found online at [35].

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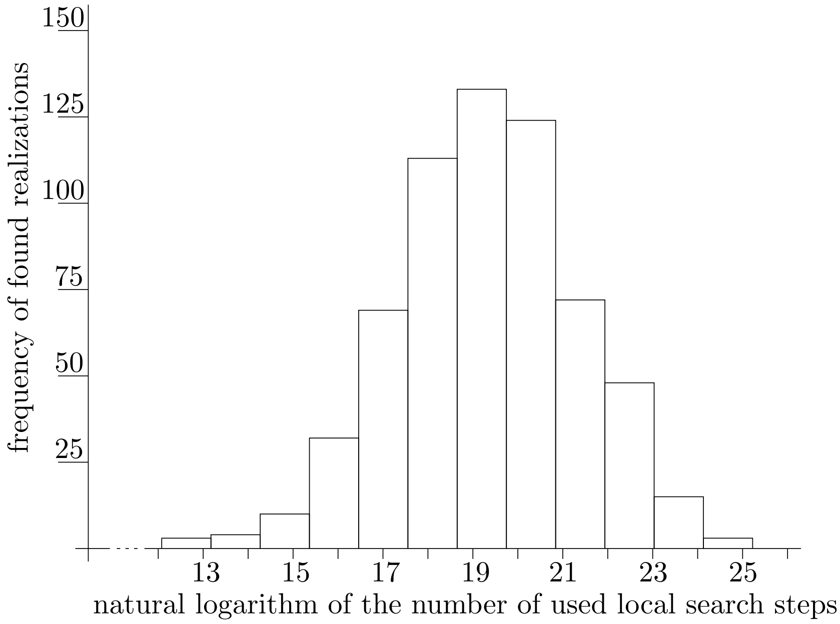

Figure 7: Histogram of the natural logarithms of the used local search steps for 626 realizations of genus 4

Genus 4

With our algorithm we found realizations for 626 of the 821 vertex-minimal triangulations of the orientable surface of genus 4. Realizations for the remaining 195 triangulations were obtained *by recycling of coordinates*, i.e., whenever a new realizations was found we tried to reuse the respective set of coordinates for other triangulations. We also slightly distorted the coordinates and then tried to use these coordinates for other triangulations; see [36] for additional comments.

Theorem 3 *All* 821 *vertex-minimal* 11*-vertex triangulations of the orientable surface of genus* 4 *are geometrically realizable in* $\mathbb{R}^3$.

We needed a total of $9.51 \cdot 10^{11}$ steps of the local search process to realize the 626 triangulations. As time interval T we chose 15 minutes, so if after 15 minutes a realization was not reached, the search was restarted with new initial coordinates.

Figure 7 displays a histogram of the natural logarithms of the number of used steps. The picture indicates that the logarithms of the used steps are normally distributed, i.e., the used steps underlie a log-normal distribution. To confirm this, we ran as a goodness-of-fit-test [50, Ch. 30] the Anderson-Darling test (cf. [25, p. 10]) to specify whether the logarithms of the used steps follow a normal distribution.

The Anderson-Darling test for a normal distribution of the natural logarithms of the used steps estimates the mean to be 19.3 and the standard deviation to be 2. It yields a p-value of 0.5 which is far above the rejection value of 0.05. Therefore we can view the logarithms of the used steps to be normally distributed with the estimated parameters.

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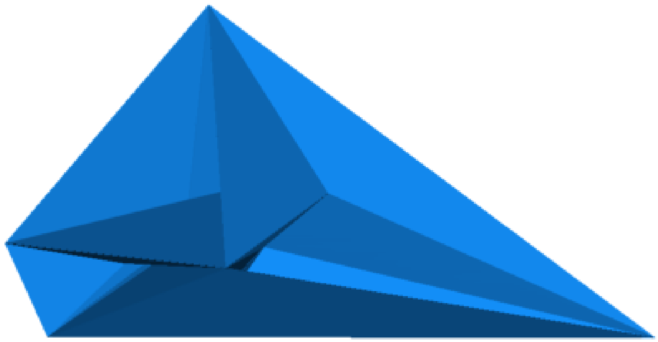

Figure 8: A polyhedron of genus 5.

Our implementation of the local search process is performing about $3.6 \cdot 10^5$ steps per minute on a 3.5 GHz processor. Therefore, we needed a total of 5 CPU years to realize all triangulations. On average, it took 2.9 CPU days for finding a realization for a single triangulation.

Genus 5

As mentioned in Section 2, Schewe [45] recently showed that there are at least three examples of vertex-minimal 12-vertex triangulations of the orientable surface of genus 5 that cannot be realized geometrically in $\mathbb{R}^3$.

In order to complement Schewe's result, we tried to find realizations for at least some of the 751.593 triangulations. To this aim we started our process on randomly selected triangulations out of all the 751.593 vertex-minimal triangulations. If after 15 minutes a realization was not found, a new triangulation was selected at random. This way, we tried about 94.000 triangulations, using a total of $7.52 \cdot 10^{11}$ local search steps – a CPU time of approximately 4 years – and succeeded in realizing 15 triangulations.

Theorem 4 *At least* 15 *of the* 751.593 *vertex-minimal* 12*-vertex triangulations of the orientable surface of genus* 5 *are geometrically realizable in* $\mathbb{R}^3$*.*

Example 4: Figure 8 displays one of the polyhedra of genus 5 with 12 vertices, which has triangles

123	124	135	146	157	168	179	18 10	19 10	236
245	258	26 10	28 11	29 11	29 12	2 10 12	35 11	368	378
37 10	39 10	39 11	459	46 11	478	47 12	489	4 10 11	4 10 12
569	56 10	57 10	58 12	5 11 12	679	67 12	6 11 12	89 12	8 10 11

and coordinates

1: (137,124,141)	2: (107,118,143)	3: (132,130,125)	4: (122,127,129)
5: (124,129,132)	6: (126,130,124)	7: (126,129,129)	8: (122,125,138)
9: (124,128,136)	10: (119,133,134)	11: (120,130,135)	12: (121,128,133).

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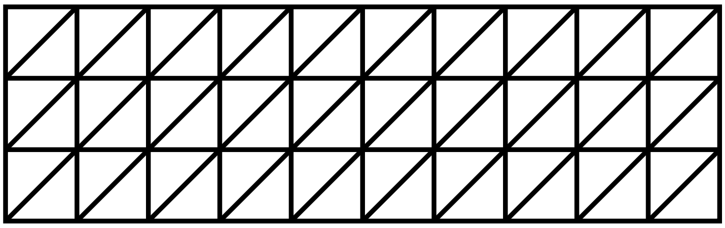

Figure 9: The standard (3×10)-torus.

The coordinates for the other 14 examples can be found online at [35].

Combining the result of Schewe [45] (that there are non-realizable triangulations of the orientable surface of genus 5) with our finding (that all vertex-minimal triangulations of surfaces of genus $g \leq 4$ are realizable) gives rise to:

Conjecture 5 *Every triangulation of an orientable surface of genus $g \leq 4$ is geometrically realizable.*

4.1 Examples with More Vertices

We also tried our program on some triangulations of tori with more vertices. It turned out that it still is possible to find realizations, although it takes much longer for every step of the local search process: There are $O(|V|^2)$ pairs of triangles that have to be considered for the computation, respectively for the update, of the intersection edge functional. Moreover, there are $6|V|$ possible moves from a current set of coordinates that lead to a new set of coordinates. In the worst case, we are forced to test almost all these moves just to carry out a single improvement step. Finally, the initial value of the intersection edge functional will be larger for triangulations with more vertices, thus, forcing us to perform more steps.

Example 5: For the standard (3×10)-torus (Figure 9) we started with random coordinates (Figure 10, left) and an initial value 7924.26 of the intersection edge functional. It then took 3042 local search steps to obtain a proper realization, as given in Figure 10 at the right.

5 Convex Realizations

If a set of n points in general position in $\mathbb{R}^3$ allows for a *convex realization* of a triangulated 2-sphere with n vertices, then there are no intersections between a face of the triangulation with 1, 2, or 3 elements and a non-face with 2, 3, or 4 elements with respect to the given coordinates.

Thus, by adding to the intersection edge functional the lengths of intersection edges for all pairs of triangles consisting of a triangle of the triangulation and a triangle that does not belong to the triangulation the resulting *extended intersection edge functional* can be used to obtain convex realizations for triangulated 2-spheres.

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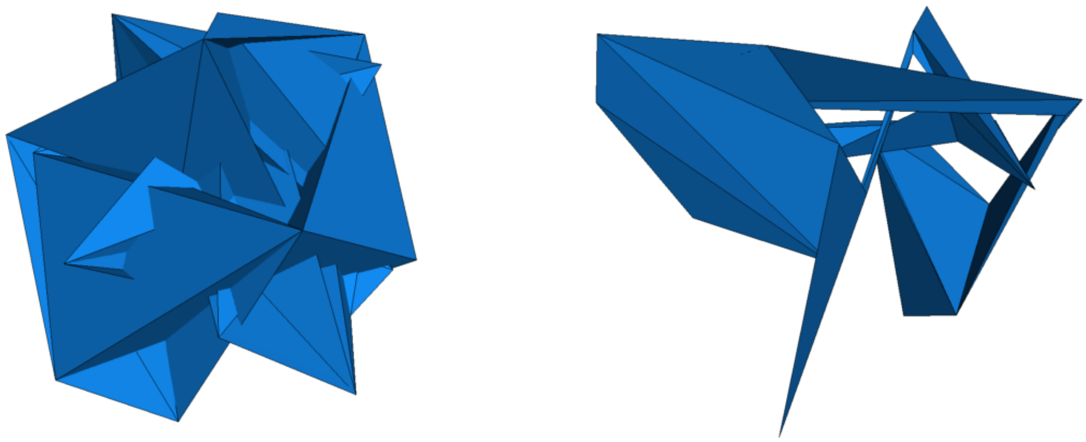

Figure 10: The standard (3×10)-torus with random coordinates and a proper realization.

Proposition 6 *If a triangulated 2-sphere has no vertex of degree 3, then the extended intersection edge functional is zero if and only if a convex realization (with vertices in general positions) has been reached.*

Proof. If a convex realization has been reached, then obviously the extended intersection edge functional is zero.

For the other direction, assume that the functional is zero. In case there is a vertex v that is contained in the convex hull of the other $n-1$ vertices, then this vertex is contained in the convex hull of some subset $\{v_1, v_2, v_3, v_4\}$ of four of the $n-1$ vertices. Without loss of generality, we can assume that no other of the $n-1$ vertices is contained in the convex hull of the four vertices $\{v_1, v_2, v_3, v_4\}$. Since the vertex v has degree at least 4, at least one of its triangles has to intersect the boundary of the tetrahedron spanned by the vertices v_1, v_2, v_3, v_4, contradiction. Thus, the n vertices have to be in convex position. If there is a triangle $\{w_1, w_2, w_3\}$ of the triangulation which is not a boundary triangle of the convex polytope spanned by the n vertices, then there are two vertices w_4 and w_5 of the triangulation which lie on different sides of the hyperplane spanned by the triangle $\{w_1, w_2, w_3\}$. Since the five points w_1, w_2, w_3, w_4, w_5 are in convex position, there is one vertex w_i, $i \in \{1, 2, 3\}$, such that the triangles $\{w_1, w_2, w_3\}$ and $\{w_i, w_4, w_5\}$ intersect, contradiction. □

In case a triangulation has vertices of degree 3, we can recursively remove these vertices from the triangulation. The resulting triangulation then either is the boundary of a tetrahedron or a triangulation with vertices of degree at least four. After obtaining a realization for the simplified triangulation, the removed vertices can be added back by placing them suitably above the triangles which they subdivide.

We successfully tested our approach for some smaller triangulations of S^2: there are 233 triangulations of S^2 with 10 vertices of which 12 have no vertices of degree 3. It took, on average, about 5 minutes to obtain convex realizations for these examples.

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Remark: Not all simplicial 3-spheres are polytopal. (The Brückner-Grünbaum sphere [28] and the Barnette sphere [5], both with 8-vertices, are the smallest non-polytopal simplicial 3-spheres.)

However, for a given simplicial 3-sphere one might try to carry out the same procedure as above in order to obtain a convex realization for it in $\mathbb{R}^4$, this time by using an *intersection area functional.*

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