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Note

Counterexamples to three conjectures concerning perfect graphs

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Abstract We will present counterexamples to a conjecture of Hoàng, a conjecture of Hertz and de Werra and to a conjecture of Reed. All these three conjectures are related to perfect graphs.

1. Introduction

A graph G is called *perfect* iff for every induced subgraph $H \subseteq G$ the chromatic number of H equals the size of a largest clique in H. A graph is called *Berge* iff neither G nor $\overline{G}$ contains an odd induced cycle of length greater than three. In 1960 Berge 2 posed the following conjecture which is still open:

The Strong Perfect Graph Conjecture *A graph is perfect iff it is Berge.*

This conjecture (called SPGC for short) has motivated several people to work on perfect graphs and many partial results were obtained. Probably the most important result is the following theorem of Lovász 15 which was conjectured by Berge together with the SPGC:

The Perfect Graph Theorem *A graph is perfect iff its complement is perfect.*

In the following three sections we will present counterexamples to three conjectures concerning perfect graphs.

2. A conjecture on alternately orientable graphs

One of the first classes of graphs which has been shown to be perfect is the class of transitively orientable graphs (also called comparability graphs). A graph G is called *transitively orientable* if the edges of G can be oriented in such a way that

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no induced directed P_3 exists. The perfection of transitively orientable graphs has been established by Berge 1. An orientation of a graph is called *acyclic* if no directed cycle exists. Ghouila-Houri 7 proved the following statement:

Theorem *A graph admits a transitive orientation iff it admits an orientation that is transitive and acyclic.*

A graph is called *alternately orientable* if it admits an orientation of the edges that alternates on every induced cycle of length greater than three. Immediately from the definition it follows that every transitively orientable graph is alternately orientable. The class of alternately orientable graphs was introduced by Hoàng 11 who proved that this class of graphs is perfect. In the same paper he conjectured that a statement similar to the above theorem of Ghouila-Houri might hold:

Conjecture 1 *A graph admits an alternating orientation iff it admits an orientation that is alternating and acyclic.*

The following graph G is a counterexample to this conjecture:

Figure 1 goes around here

This graph is alternately orientable but in every alternating orientation the triangle *abc* will get a cyclic orientation. To see this regard the following graph H:

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Figure 2 goes around here

It is easy to see that this graph is alternately orientable and that the edge ac may be oriented arbitrarily in every such orientation while the edges ab and bc form a directed P_3. (The arrows in the above picture show one of the two possible orientations of those edges which lie in an induced cycle of length greater than three. The unoriented edges may be oriented arbitrarily since they lie in no such cycle.) Let H' be a copy of H and a', b' and c' the copies of a, b and c. The graph G was obtained from H and H' by identifying a' with b, b' with c and c' with a. Since H and H' are identified in a clique every induced cycle of G is completely contained in H or H'. Therefore G is also alternately orientable. But the triangle abc is now forced to have a cyclic orientation.

3. A conjecture on even pairs

A graph is called *minimal imperfect* if the graph itself is not perfect but all of its proper induced subgraphs are. Two vertices in a graph G are called an *even pair* if every induced path connecting these two vertices has even length. Fonlupt and Uhry 6 have proved that contracting an even pair in a perfect graph yields again a perfect graph. This result was essential in the proof of the following lemma of Meyniel 16:

The Even Pair Lemma *No minimal imperfect graph contains an even pair.*

For several classes of perfect graphs (e.g. weakly triangulated graphs 8, perfectly orderable graphs 10 and quasi-Meyniel graphs 9) there exist polynomial algorithms to find an optimal coloring of these graphs which are based on the contraction of even pairs. Hertz and de Werra observed that in these algorithms the two vertices of the contracted even pair are always at distance two from each other. This motivated them to pose the following conjecture 518:

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Conjecture 2 *If a graph contains an even pair then it has an even pair at distance two.*

The following graph is a counterexample to this conjecture:

Figure 3 goes around here

The two vertices a and b have distance four and they are the only even pair in the graph. To see this look at the graph which is obtained by removing the vertices a and b. This graph is a well-known even-pair-free graph 5. Thus by symmetry it remains to check that the only even pair containing the vertex a is the pair (a, b). This is easily done.

4. A conjecture concerning Berge graphs

Besides the Even Pair Lemma of Meyniel there exists another often used property of minimal imperfect graphs which was established by Chvátal in 1985 3 and uses the notion of a star-cutset. A *star-cutset* of a graph G is a cutset $\mathcal{C}$ of G that contains a vertex x that is connected to all other vertices of $\mathcal{C}$.

The Star-Cutset Lemma *No minimal imperfect graph contains a star-cutset.*

From a theorem of König 14 on bipartite graphs one can easily derive the following lemma:

Lemma *No minimal imperfect graph is the linegraph of a bipartite graph.*

These lemmas motivated Reed in 1986 517 to state the following conjecture:

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Conjecture 3 *Every Berge graph G satisfies at least one of the following conditions:*

1. *G or $\overline{G}$ contains a star-cutset;*
2. *G or $\overline{G}$ contains an even pair;*
3. *G or $\overline{G}$ is the linegraph of a bipartite graph.*

From the Star-Cutset Lemma, the Even Pair Lemma, the lemma of König and the Perfect Graph Theorem it follows that this conjecture would imply the SPGC. With the following graph we give a counterexample to Reed's conjecture:

Figure 4 goes around here

To verify that this graph is a counterexample regard the following graph:

Figure 5 goes around here

It is easy to see that this graph is Berge, contains no even pair and that all induced paths between a and b have odd length. From two lemmas in 13 it follows that G is also Berge and has no even pair. Since every two adjacent vertices in G are the

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midpoints of an induced P_4 there exists an induced path of length three between any two nonadjacent vertices in $\overline{G}$. Using a characterization of star-cutsets given by Chvátal 3 one can easily verify that neither G nor $\overline{G}$ contains a star-cutset. Since G and $\overline{G}$ contain both a $K_{1,3}$ they are both not linegraphs of a bipartite graph.

It was suggested by Chvátal 4 that one might prove the SPGC for C_4-free graphs by proving Reed's conjecture for this class of graphs. The following example shows that the conjecture of Reed remains false even for the class of C_4-free graphs:

Figure 6 goes around here

The verification of this counterexample can be done in a similar way as for the preceding one.

The lemmas in 13 mentioned above give a general method to construct counterexamples to Reed's conjecture. It was observed by Hoàng 12 that every such counterexample is diamond-free. Since the SPGC has been proved for diamond-free graphs 19 and every linegraph of a bipartite graph is diamond-free Hoàng suggested to replace the third condition in Reed's conjecture by the condition that G or $\overline{G}$ is diamond-free. We do not know whether this modification of Reed's conjecture is true.

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References

1 C. Berge, *Les problèmes de coloration en théorie des graphes,* Publ. Inst. Statist. Univ. Paris 9, 1960, 123–160.

2 C. Berge, *Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind,* Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg, 1961, 114–115.

3 V. Chvátal, *Star-Cutsets and Perfect Graphs,* J. Comb. Th. B 39, 1985, 189–199.

4 V. Chvátal, *Personal communication,* Bonn 1990.

5 C. de Figueiredo and V. Tardif, *Some Problems on Perfect Graphs,* Research Report CORR 89-31, Dept. of Combinatorics and Optimization, University of Waterloo, Ontario, August 1989.

6 J. Fonlupt and J.P. Uhry, *Transformations which Preserve Perfectness and H-Perfectness of Graphs,* Annals of Discrete Mathematics 16, 1982, 83–95.

7 A. Ghouila-Houri, *Caracterisation des graphes non orientés dont on peut orienter les aretes de manière à obtenir le graphe d'une relation d'ordre,* C.R. Acad. Sci. Paris 254, 1962, 1370–1371.

8 R.B. Hayward, C.T. Hoàng and F. Maffray, *Optimizing Weakly Triangulated Graphs,* Graphs and Combinatorics 5, 1989, 339–349. see also *Erratum*, vol. 6, 33–35, 1990

9 A. Hertz, *A Fast Algorithm for Coloring Meyniel Graphs,* Journal of Combinatorial Theory, Series B 50, 1990, 231–240.

10 A. Hertz and D. de Werra, *Perfectly Orderable Graphs are Quasi-Parity Graphs: A Short Proof,* Discrete Mathematics 68, 1988, 111–113.

11 C.T. Hoàng, *Alternating Orientation and Alternating Colouration of Perfect Graphs,* J. Comb. Th. B 42, 1987, 264–273.

12 C.T. Hoàng, *Personal communication,* Bonn 1991.

13 S. Hougardy, *Perfekte Graphen,* Diplomarbeit, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany, 1991.

14 D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre,* Mathematische Annalen 77,1916, 453–465.

15 L. Lovász, *A Characterization of Perfect Graphs,* J. Comb. Th. B 13, 1972, 95–98.

16 H. Meyniel, *A new Property of Critical Imperfect Graphs and some Consequences,* Europ. J. Combinatorics 8, 1987, 313–316.

17 B. Reed, *A Semi-Strong Perfect Graph Theorem,* Ph. D. Thesis, Mc Gill University, Québec 1986.

18 B. Reed, *Perfection, Parity, Planarity, and Packing Paths,* Integer Programming and Combinatorial Optimization, Kannan and Pulleyblank eds., University of Waterloo Press 1990.

19 A. Tucker, *Coloring Perfect $(K_4 - e)$-Free Graphs,* J. Comb. Th. B 42, 1987, 313–318.