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On the Integrality Ratio of the Subtour LP for Euclidean TSP

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Abstract

A long standing conjecture says that the integrality ratio of the subtour LP for metric TSP is 4/3. A well known family of graphic TSP instances achieves this lower bound asymptotically. For Euclidean TSP the best known lower bound on the integrality ratio was 8/7. We improve this value by presenting a family of Euclidean TSP instances for which the integrality ratio of the subtour LP converges to 4/3.

keywords: traveling salesman problem; subtour LP; Held-Karp bound; integrality ratio; Euclidean TSP

1 Introduction

Given n cities with their pairwise distances the *traveling salesman problem* (TSP) asks for a shortest tour that visits each city exactly once. This problem is known to be NP-hard [6] and cannot be approximated within any constant factor unless P=NP [9]. For *metric TSP* instances, i.e., TSP instances where the distances satisfy the triangle inequality, a 3/2 approximation algorithm was presented by Christofides in 1976 [1]. In spite of many efforts no improvement on this approximation ratio for the general metric TSP has been achieved so far. One approach to obtain a better approximation algorithm for the metric TSP is based on the *subtour LP*. This LP is a relaxation of an integer program for the TSP that was first used by Dantzig, Fulkerson, and Johnson in 1954 [2]. If the cities are numbered from 1 to n and c_{ij} denotes the distance between city i and city j then the subtour LP can be formulated as follows:

$$\text{minimize:} \quad \sum_{1 \leq i < j \leq n} c_{ij} \cdot x_{ij}$$

subject to:

$$\sum_{j \neq i} x_{ij} \;=\; 1 \quad \text{for all } i \in \{1, \ldots, n\}$$

$$\sum_{j \neq i} x_{ji} \;=\; 1 \quad \text{for all } i \in \{1, \ldots, n\}$$

$$\sum_{i,j \in S} x_{ij} \;\leq\; |S| - 1 \quad \text{for all } \emptyset \neq S \subsetneq \{1, \ldots, n\}$$

$$0 \;\leq\; x_{ij} \;\leq\; 1$$

This LP has an exponential number of constraints but can be solved in polynomial time via the ellipsoid method as the separation problem can be solved efficiently [5]. The *integrality ratio* of the subtour LP for the metric TSP is the supremum of the length of an optimum TSP tour over the optimum solution of the subtour LP. Wolsey [11] has shown that the integrality ratio of the subtour LP for metric TSP is at most $3/2$. A well known conjecture states that the integrality ratio of the subtour LP for metric TSP is $4/3$. This conjecture seems to be mentioned for the first in 1990 [10, page 35] but according to [4] it was already well known several years before. A proof of this conjecture yields a polynomial time algorithm that approximates the value of an optimum TSP tour within a factor of $4/3$. It is known that the integrality ratio of the subtour LP is at least $4/3$ as there exists a family of metric TSP instances whose integrality ratio converges to $4/3$ (see for example [10]).

For the metric TSP the lower and upper bound of $4/3$ and $3/2$ on the integrality ratio of the subtour LP have not been improved for more than 25 years. Therefore, people became interested to study the integrality ratio of the subtour LP for special cases of the metric TSP.

The *graphic TSP* is a special case of the metric TSP where the distances between the n cities are the lengths of shortest paths in an undirected connected graph on these n cities. For graphic TSP the integrality ratio of the subtour LP is at least $4/3$ [10] and at most 1.4 [7].

In the 1-2-TSP the distances between two cities are either 1 or 2. Here the largest known lower bound on the integrality ratio of the subtour LP is $10/9$ [10] while the smallest known upper bound is $5/4$ [8].

In the *Euclidean TSP* the cities are points in the plane and their distance is the Euclidean distance between the two points. In this case the best known lower bound on the integrality ratio seems to be $8/7$ (mentioned in [11]). The best known upper bound

is as in the general metric case $3/2$.

In this paper we will improve the lower bound for the integrality ratio of the Euclidean TSP by presenting a family of Euclidean TSP instances for which the integrality ratio of the subtour LP converges to $4/3$. We will prove this result in Section 2. Using a more careful analysis we prove an explicit formula for the integrality ratio in Section 3.

2 Euclidean instances with integrality ratio 4/3

We will now describe our construction of a family of Euclidean TSP instances for which the integrality ratio of the subtour LP converges to $4/3$. Each instance of the family contains equidistant points on three parallel lines. More precisely, it contains the $3n$ points with coordinates $(i, j \cdot d)$ for $i = 1, \ldots, n$ and $j = 1, 2, 3$ where d is the distance between the parallel lines. We will denote these instances by $G(n, d)$. Figure 1 shows the instance $G(18, 3)$.
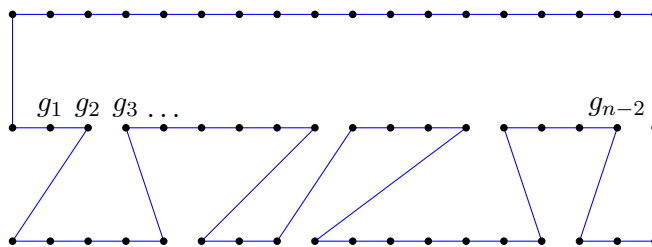

Figure 1: A TSP tour for the instance $G(18, 3)$.

The instances $G(n, d)$ belong to the class of so called *convex-hull-and-line TSP*. These are Euclidean TSP instances where all points that do not lie on the boundary of the convex hull of the point set lie on a single line segment inside the convex hull. Deineko, van Dal, and Rote [3] have shown that an optimum TSP tour for a convex-hull-and-line TSP can be found in polynomial time.

Following the notation in [3] we denote the point $(i+1, 2 \cdot d)$ in the instance $G(n, d)$ by g_i for $i = 1, \ldots, n-2$ (see Figure 1). Thus, the set $\mathcal{G} := \{g_1, \ldots, g_{n-2}\}$ contains all points of the instance $G(n, d)$ that do not lie on the boundary of the convex hull of $G(n, d)$. Let $\mathcal{B}$ denote the set of all other points in $G(n, d)$. An optimum TSP tour for $\mathcal{B}$ is obtained by visiting the points in $\mathcal{B}$ in their cyclic order [3]. Moreover, this tour is unique. Therefore, we can call two points in $\mathcal{B}$ *adjacent* if they are adjacent in the optimum tour for $\mathcal{B}$. Let $\mathcal{B}_l$ denote all points in $\mathcal{B}$ that lie on the two lower lines. The following structural result is a special case of Lemma 3 in [3]:

Lemma 1 (Lemma 3 in [3]) *An optimum TSP tour for the instance $G(n, d)$ can be obtained by splitting the set of points $\mathcal{G}$ into $k + 1$ segments*

$$\{g_1, g_2, \ldots, g_{i_1}\}, \quad \{g_{i_1+1}, \ldots, g_{i_2}\}, \quad \ldots, \quad \{g_{i_k+1}, \ldots, g_{n-2}\}$$

for $0 \leq k \leq n - 2$, $0 = i_0 < i_1 < i_2 < \cdots < i_k < n - 2$, and inserting each segment between two adjacent points in $\mathcal{B}_l$.

Proof. This is exactly the statement of Lemma 3 in [3] except that $\mathcal{B}$ is replaced by $\mathcal{B}_l$. Thus, we only have to observe that because of symmetry of the instance $G(n,d)$ we may assume that all segments are inserted into adjacent points in $\mathcal{B}_l$. $\square$

Let the *cost* of inserting a segment $\{g_i, g_{i+1}, \ldots, g_j\}$ into $\mathcal{B}$ be the difference in the length of the optimum tour after and before inserting this segment. From Lemma 4 in [3] it follows that a segment $\{g_i, g_{i+1}, \ldots, g_j\}$ that contains neither g_1 nor g_{n-2} must be inserted between two adjacent points from the lower line of points in $G(n,d)$. As the points have unit distance on the lines, the cost of inserting such a segment only depends on the number of points contained in the segment.

Lemma 2 *For $d \geq 4$ the cost of inserting a segment of i points from $\mathcal{G}$ into $\mathcal{B}$ is at least*

$$\begin{cases} i - 2 + \max\{2d, i - 2\} & \text{if } g_1 \text{ and } g_{n-2} \text{ do not belong to the segment} \\ i - d + \max\{d, i\} & \text{if } g_1 \text{ or } g_{n-2} \text{ belong to the segment} \end{cases}$$

Proof. As already mentioned above it follows from Lemma 4 in [3] that a segment that contains neither g_1 nor g_{n-2} must be inserted between two adjacent points from the lower line of points in $G(n,d)$. The total cost of inserting such a segment is $i - 1$ for the horizontal connection of the i points in the segment plus the two connections from the end points of the segment to two adjacent points in the lower line of points in $G(n,d)$ minus 1 (for the edge that is removed between the two adjacent points in $\mathcal{B}$). Thus we get a lower bound of $i - 2 + \max\{2d, i - 2\}$.

If g_1 or g_{n-2} is contained in the segment then there are two possibilities to insert the segment which are shown in Figure 2. For case b) we get again the lower bound $i - 2 + \max\{2d, i - 2\}$ while in case a) we have a lower bound of $i - d + \max\{d, i\}$. Since $i - 2 + \max\{2d, i - 2\} \geq i - d + \max\{d, i\}$ for $d \geq 4$ the result follows. $\square$
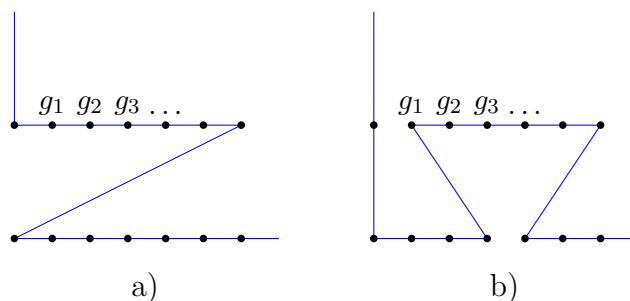

Figure 2: The two possibilities to insert into $\mathcal{B}$ a segment containing the point g_1.

We now can compute a lower bound on the length of an optimum TSP tour for $G(n,d)$ as follows.

Lemma 3 *Let $d \geq 4$. Then an optimum TSP tour for $G(n,d)$ has length at least $4n + 2d - 2 - 2n/(d+1)$.*

Proof. For k with $1 \leq k \leq n-2$ let $z_1, z_2, \ldots, z_k$ be the number of points contained in the segments of $\mathcal{G}$ that are inserted into $\mathcal{B}$ for an optimum TSP tour of $G(n,d)$. We have $\sum_{i=1}^{k} z_i = n-2$. The boundary of the convex hull for the point set $\mathcal{B}$ has length $2n - 2 + 4d$.

By Lemma 2 the total length of the tour is at least

$$2n - 2 + 4d + \sum_{i=1}^{k}(z_i - 2 + 2d) - 4d + 4 \quad = \quad 3n + 2k(d-1)$$

where the term $-4d + 4 \leq 0$ is for adjusting for the at most two segments containing g_1 and g_{n-2}.

On the other hand by Lemma 2 the total length of the tour is at least

$$2n - 2 + 4d + \sum_{i=1}^{k}(2z_i - 4) - 2d + 8 \quad = \quad 4n + 2 - 4k + 2d$$

Here we need the term $-2d + 8 \leq 0$ to adjust for the at most two segments containing g_1 and g_{n-2}.

This shows that an optimum TSP tour for $G(n,d)$ has length at least

$$\min_{1 \leq k \leq n-2} \max\{3n + 2k(d-1), 4(n-k) + 2d + 2\}. \tag{1}$$

The two functions $3n + 2k(d-1)$ and $4(n-k) + 2d + 2$ are both linear in k and the slopes have opposite sign. Thus, the expression (1) is at least as large as the value at the intersection of these two linear functions. The two linear functions intersect at $k = 1 + n/(2d+2)$. The value at the intersection point is $4(n - 1 - n/(2d+2)) + 2d + 2 = 4n + 2d - 2 - 2n/(d+1)$. This finishes the proof. $\square$

Now we can state and prove our main result.

Theorem 4 *Let d be a function with $d(n) = \omega(1)$ and $d(n) = o(n)$. Then the integrality ratio of the subtour LP for the instances $G(n, d(n))$ converges to $4/3$ for $n \to \infty$.*

Proof. As $d(n) = \omega(1)$ we may assume that $d(n) \geq 4$. By Lemma 3 the length of any TSP tour for $G(n, d(n))$ is at least $4n + 2d - 2 - 2n/(d+1)$.

Figure 3 shows a feasible solution for the subtour LP for $G(n, d(n))$ with cost $3n - 4 + 3d(n) + \sqrt{d(n)^2 + 1} \leq 3n + 4d(n)$. Thus, the integrality ratio for $G(n, d(n))$ is at least

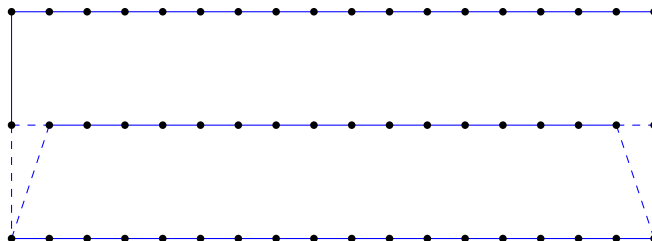$$\frac{4n + 2d(n) - 2 - 2n/(d(n) + 1)}{3n + 4d(n)} \tag{2}$$

Figure 3: A feasible solution to the subtour LP for the instance $G(18, 3)$. The dashed lines correspond to variables with value $1/2$ while all other lines correspond to variables with value 1.

which tends to $4/3$ for $n \to \infty$ as $d(n) = \omega(1)$ and $d(n) = o(n)$. □

The proofs of Lemma 3 and Lemma 4 in [3] make only use of the fact that optimum TSP tours for certain subsets of the point set do not intersect itself. This property holds in case of the instance $G(n, d)$ for all L^p-norms with $p \in \mathbb{N}$. Moreover, the proofs of Lemma 2 and Lemma 3 also hold for arbitrary L^p-norms. Therefore, Theorem 4 holds for all L^p-norms with $p \in \mathbb{N}$, as $3n + 4d(n)$ is an upper bound for an optimum solution for the subtour LP for any L^p-norm .

3   The exact integrality ratio for $G(n, \sqrt{n-1})$

The lower bound (2) proven in Section 2 attains its maximum for $d(n) = \Theta(\sqrt{n})$. In this section we prove an explicit formula for the integrality ratio of the instances $G(n, \sqrt{n-1})$. This leads to an improved convergence of the proven lower bound. The structural results for an optimum TSP tour in the instances $G(n, d)$ proven in this section may be of independent interest.

Lemma 5 *For the instance $G(n, d)$ the cheapest cost of inserting into $\mathcal{B}$ a segment of k points of $\mathcal{G}$ that contains neither g_1 nor g_{n-2} is*

$$
\begin{cases}
k - 2 + \sqrt{(k-2)^2 + 4 \cdot d^2} & \text{if } k \text{ is even} \\
k - 2 + \frac{1}{2} \cdot \left(\sqrt{(k-1)^2 + 4 \cdot d^2} + \sqrt{(k-3)^2 + 4 \cdot d^2} \right) & \text{if } k \text{ is odd}
\end{cases}
$$

Proof. By shifting the instance $G(n, d)$ we may assume that the k points in the segment have the coordinates (i, d) for $i = 1, \ldots, k$. The cost of inserting this segment between the two points $(a, 0)$ and $(a+1, 0)$ with $a \in \{1, \ldots, k\}$ is $(k-1) + \sqrt{(a-1)^2 + d^2} + \sqrt{(k-a-1)^2 + d^2} - 1$. The minimum is attained for $a = \lfloor \frac{k}{2} \rfloor$. □

If a segment of $\mathcal{G}$ contains g_1 or g_{n-2} then there exist two different possibilities for inserting it into $\mathcal{B}$, see Figure 2. The following lemma states, that if d is sufficiently large

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then possibility a) will be cheaper. For the proof of this lemma and in some other proofs we will make use of the inequality

$$a + \sqrt{b^2 + c} \geq \sqrt{(a+b)^2 + c} \qquad \text{for all } a, b, c \in \mathbb{R}_+ \tag{3}$$

Lemma 6 *Let $d \geq 4$. For the instance $G(n, d)$ the cheapest cost of inserting into $\mathcal{B}$ a segment of k points of $\mathcal{G}$ that contains either g_1 or g_{n-2} is*

$$k - d + \sqrt{k^2 + d^2}.$$

Proof. There exist two different possibilities to insert into $\mathcal{B}$ a segment of k points of $\mathcal{G}$ that contains the point g_1. These two possibilities are shown in Figure 2a) and b). In case a) the cost of inserting the segment is $k - d + \sqrt{k^2 + d^2}$. In case b) the cost of insertion is by Lemma 5 at least $k - 2 + \sqrt{(k-2)^2 + 4 \cdot d^2}$. We claim that for $d \geq 4$ we have $k - d + \sqrt{k^2 + d^2} \leq k - 2 + \sqrt{(k-2)^2 + 4 \cdot d^2}$. This is equivalent to the statement that for $d \geq 4$ we have

$$h(d) := d - 2 + \sqrt{(k-2)^2 + 4 \cdot d^2} - \sqrt{k^2 + d^2} \geq 0.$$

Now, $h(4) = 2 + \sqrt{(k-2)^2 + 64} - \sqrt{k^2 + 16} \geq \sqrt{k^2 + 64} - \sqrt{k^2 + 16} \geq 0$ by (3). Moreover, we have $h'(d) = \frac{4d}{\sqrt{4d^2 + (k-2)^2}} - \frac{d}{\sqrt{d^2 + k^2}} + 1 > 0$ which proves the claim. $\square$

From Lemma 5 and Lemma 6 it follows that for $d \geq 4$ there always exists an optimum tour of $G(n, d)$ which has a *z-structure* as shown in Figure 4. This means that the tour consists of a sequence of alternately oriented z-shaped paths that cover all points in the lower two lines. These z-shaped paths are connected by single edges of length 1 and the tour is closed by adding the top most horizontal line and two vertical connections to the middle line. Each tour with such a z-structure can be specified by a *z-vector* which contains as entries the number of points covered by each z on the middle line.
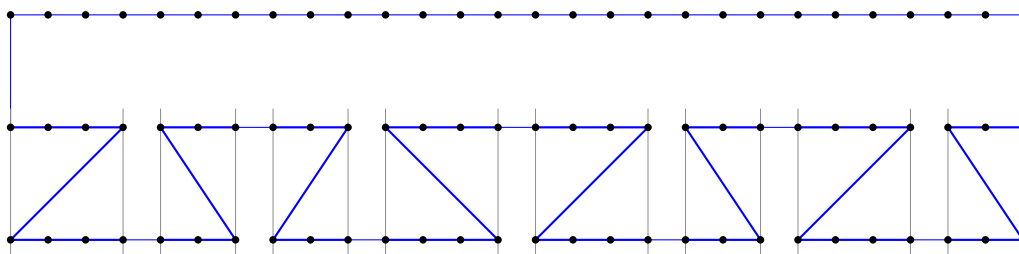

Figure 4: The z-structure of a TSP tour for the instance $G(28, 3)$. The tour shown has the z-vector $(4, 3, 3, 4, 4, 3, 4, 3)$.

The length of a tour can now easily be computed using its z-vector. Set

$$c(i) := 2(i - 1) + \sqrt{(i-1)^2 + d^2}.$$

Then $c(i)$ is the length of a z-shaped path covering i points in the middle row.

Lemma 7 *Let $d \geq 4$ and $k \geq 2$. Then the total length of a TSP tour for $G(n, d)$ corresponding to a z-vector $(z_1, z_2, \ldots, z_k)$ is*

$$n + k + 2d - 2 + \sum_{i=1}^{k} c(z_i).$$

Proof. The length of all the z-shaped subpaths of the tour is $\sum_{i=1}^{k} c(z_i)$. These k subpaths are connected by $k - 1$ edges of length 1. In addition there is one path of length $n - 1$ connecting all the points in the upper row plus two vertical edges of length d each which connect the upper row with the middle row. □

The next result shows that there always exists an optimum TSP tour for $G(n, d)$ which has a very special structure.

Lemma 8 *Let $d \geq 4$, $k \geq 2$ and $(z_1, z_2, \ldots, z_k)$ be the z-vector of an optimum TSP tour for $G(n, d)$. Then $z_i \in \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$.*

Proof. Suppose this is not the case. Then there exist $i, j \in \{1, \ldots, k\}$ with $z_i \geq z_j + 2$. As the function c is convex we have $c(z_i - 1) + c(z_j + 1) < c(z_i) + c(z_j)$. By Lemma 7 this implies that $(z_1, z_2, \ldots, z_k)$ cannot be the z-vector of an optimum TSP tour for $G(n, d)$. □

The following result shows how to bound the length of a TSP tour for $G(n, d)$ with given z-vector solely by the length of the z-vector.

Lemma 9 *Let $d \geq 4$ and $k \geq 2$. Then the length of an optimum TSP tour for $G(n, d)$ corresponding to a z-vector $(z_1, z_2, \ldots, z_k)$ is at least*

$$n + k + 2d - 2 + k \cdot c(\frac{n}{k}) \ .$$

Proof. If $(z_1, z_2, \ldots, z_k)$ is an optimum TSP tour for $G(n, d)$ then we may assume that $z_1 \leq z_2 \leq \ldots \leq z_k$ as reordering the z_i-values does not change the length of the tour. Together with Lemma 8 this shows that there exists an optimum TSP tour for $G(n, d)$ which corresponds to the z-vector $z_1, \ldots, z_j, z_{j+1}, \ldots, z_k$ with $z_1 = \ldots = z_j = \lfloor n/k \rfloor$, $z_{j+1} = \ldots = z_k = \lceil n/k \rceil$, and $\sum_{i=1}^{k} z_i = n$. As c is convex we have $\sum_{i=1}^{k} c(z_i) \geq k \cdot c(\frac{n}{k})$. □

The next result shows that there always exists an optimum TSP tour for $G(n, d)$ which corresponds to an even length z-vector.

Lemma 10 *Let $d \geq 4$ and n be even. Then there exists an optimum TSP tour for $G(n, d)$ which corresponds to an even length z-vector. Moreover, the length of an optimum TSP tour for $G(n, d)$ is at least:*

$$\min_{1 \leq k \leq n/2} n + 2k + 2d - 2 + 2k \cdot c(\frac{n}{2k}) \ .$$

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Proof. By Lemma 9 we only have to rule out that there can exist an optimum TSP tour for $G(n,d)$ which corresponds to a z-vector of odd length. By Lemma 6 and its proof we know that if the length of the z-vector is odd, then it must be 1. A TSP tour for $G(n,d)$ corresponding to a z-vector of length one has length

$$3d + 3n - 4 + \sqrt{(n-2)^2 + d^2} \tag{4}$$

A tour corresponding to a z-vector of length 2 has length

$$2d + 3n - 4 + 2\sqrt{\left(\frac{n}{2} - 1\right)^2 + d^2} \tag{5}$$

Now using (3) we have

$$d + \sqrt{(n-2)^2 + d^2} \geq \sqrt{(n-2)^2 + 4d^2} = \sqrt{\left(\frac{n}{2} - 1\right)^2 + d^2} .$$

This implies (4)$\geq$(5) and therefore the minimum is always attained for an even length z-vector. $\qquad\square$

Theorem 11 *Let $n \geq 17$ be even. Then an optimum TSP tour for $G(n, \sqrt{n-1})$ has length $4n - 4 + 2\sqrt{n-1}$.*

Proof. Let

$$
\begin{aligned}
f(k,d) \quad &:= \quad n + 2k + 2d - 2 + 2k \cdot c\left(\frac{n}{2k}\right) \\
&= \quad n + 2k + 2d - 2 + 2k \cdot \left(2 \cdot \left(\frac{n}{2k} - 1\right) + \sqrt{\left(\frac{n}{2k} - 1\right)^2 + d^2}\right) \\
&= \quad 3n - 2k + 2d - 2 + \sqrt{(n - 2k)^2 + 4d^2k^2}
\end{aligned}
$$

For $d = \sqrt{n-1}$ we get

$$
\begin{aligned}
f(k, \sqrt{n-1}) \quad &= \quad 3n - 2k + 2\sqrt{n-1} - 2 + \sqrt{(n-2k)^2 + 4(n-1)k^2} \\
&= \quad 3n - 2k + 2\sqrt{n-1} - 2 + \sqrt{n^2 + 4nk(k-1)}
\end{aligned}
$$

From Lemma 10 and its proof we know that there exists a TSP tour in $G(n, \sqrt{n-1})$ of length $f(1, \sqrt{n-1})$. We now claim that

$$f(k, \sqrt{n-1}) > f(1, \sqrt{n-1}) \quad \text{for } k = 2, 3, \ldots, n/2. \tag{6}$$

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By Lemma 10 this finishes the proof. To prove (6) let $g(k) := f(k, \sqrt{n-1})$. We will show that $g'(k) > 0$ for $k > 1$. We have

$$g'(k) = \frac{4kn - 2n - 2\sqrt{n^2 + 4nk(k-1)}}{\sqrt{n^2 + 4nk(k-1)}}$$

Now we have

$$
\begin{aligned}
(k-1)n &\geq k-1 \\
\Rightarrow \quad k^2n^2 - kn^2 &\geq nk^2 - nk \\
\Rightarrow \quad 4k^2n^2 + n^2 - 4kn^2 &\geq n^2 + 4nk^2 - 4nk \\
\Rightarrow \quad 2kn - n &\geq \sqrt{n^2 + 4nk(k-1)}
\end{aligned}
$$

This proves that $g'(k) \geq 0$ for $k \geq 1$. $\qquad \square$

Lemma 12 *The optimum solution of the subtour LP for the instance $G(n, \sqrt{n-1})$ has value $3n - 3 + 3\sqrt{n-1} + \sqrt{n}$.*

Proof. An optimum solution to the subtour LP is indicated in Figure 3. It is easily verified that the value is as claimed. $\qquad \square$

Now we can state and prove an explicit formula for the integrality ratio of the instances $G(n, \sqrt{n-1})$.

Theorem 13 *Let $n \geq 17$ be even. The integrality ratio of the subtour LP for the instance $G(n, \sqrt{n-1})$ is*

$$\frac{4n - 4 + 2\sqrt{n-1}}{3n - 4 + 3\sqrt{n-1} + \sqrt{n}} \ .$$

Proof. This result immediately follows from Theorem 11 and Lemma 12. $\qquad \square$

Using Theorem 13 one obtains for example that the integrality ratio of $G(18, \sqrt{17})$ is 1.14. This is much larger than the lower bound 1.01 which can be obtained from (2).

The approach used to prove Theorem 13 can also be applied to other distance functions $d(n)$. Choosing $d(n) = \sqrt{n/2 - 1}$ one obtains an integrality ratio of

$$\frac{4n - 6 + 2\sqrt{n/2 - 1}}{3n - 4 + 3\sqrt{n/2 - 1} + \sqrt{n/2}} \ .$$

This value is larger than the value stated in Theorem 13. However, the proof gets a bit more complicated as now one has to prove that the function $f(k, \sqrt{n/2 - 1})$ appearing in the proof of Theorem 11 attains its minimum for $k = 2$.

Acknowledgement

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