Linear and Integer Optimization

Exercise Sheet 11

Exercise 11.1:
Let $G = (V, E)$ be a digraph with $|V| = n$ and $|E| = m$. The feasible solutions to the *directed traveling salesman problem* are defined by the following constraints:

$$\begin{aligned} \sum_{v:(v,w)\in E} x_{(v,w)} &= 1, \quad \forall\, w \in V \\ \sum_{w:(v,w)\in E} x_{(v,w)} &= 1, \quad \forall\, v \in V \\ \sum_{(v,w)\in E\,:\, v\in S, w\notin S} x_{(v,w)} &\geq 1, \quad \forall\, S \subset V, S \neq \emptyset, V \\ x_{(v,w)} &\in \{0,1\} \quad \forall\, (v,w) \in E \end{aligned}$$

Let $\mathcal{F}$ be the set of feasible solutions.

1. Now choose one vertex $v_0 \in V$ arbitrarily and consider the set $\mathcal{F}'$ of solutions to the following (polynomially many) constraints:

$$\begin{aligned} \sum_{v:(v,w)\in E} x_{(v,w)} &= 1, \quad \forall\, w \in V \\ \sum_{w:(v,w)\in E} x_{(v,w)} &= 1, \quad \forall\, v \in V \\ y_v - y_w + n x_{(v,w)} &\leq n - 1, \quad \forall\, (v,w) \in E, v, w \neq v_0 \\ x_{(v,w)} &\in \{0,1\}, \quad \forall\, (v,w) \in E \\ y_v &\in \mathbb{R}, \quad \forall\, v \in V \end{aligned}$$

   Prove: $\mathcal{F}' = \mathcal{F}$. (4 Points)

2. Let $LP_{\mathcal{F}}$ and $LP_{\mathcal{F}'}$ be the linear relaxations of $\mathcal{F}$ respectively $\mathcal{F}'$, i.e. relaxing $x_{(v,w)} \in \{0,1\}$. Show that $LP_{\mathcal{F}} \subseteq LP_{\mathcal{F}'}$ and give an example for which $LP_{\mathcal{F}} \neq LP_{\mathcal{F}'}$. (4 Points)

3. Prove or disprove: $LP_{\mathcal{F}} = \text{conv}(\mathcal{F})$. (2 Points)

Exercise 11.2: Let A be an integral $m \times n$-matrix, and let b and c be vectors, and y an optimum solution of $\max\{cx : Ax \leq b, x \text{ integral}\}$. Prove that there exists an optimum solution z of $\max\{cx : Ax \leq b\}$ with $||y - z||_\infty \leq n\Xi(A)$.

(5 Points)

Exercise 11.3: Prove that there is a polynomial-time algorithm which, given an integral matrix A and an integral vector b, finds an integral vector x with $Ax = b$ or decides that none exists.

(5 Points)

Submission deadline: Tuesday, 14.1.2014, before the lecture.