Linear and Integer Optimization

Exercise Sheet 9

Exercise 9.1: Show that for each $K \in \mathbb{N}$ there is a bounded integer program with two variables such that the branch-&-bound algorithm visits more than K vertices in the branch-&-bound tree. (5 Points).

Exercise 9.2: Consider the following ILP (integer linear program)

$$\begin{array}{ll} \max -\sqrt{2}x + y & \\ -\sqrt{2}x + y & \leq 0 \\ x & \geq 1 \\ y & \geq 0 \\ x, y & \in \mathbb{Z} \end{array}$$

Show:

1. The ILP has feasible solutions. (1 Point)
2. The objective is bounded from above by 0. (1 Point)
3. There is no optimum solution. (2 Points)
4. Let S be the set of solutions of the ILP, then $\text{conv}(S)$ is not a polyhedron. (1 Point)

Exercise 9.3: (Mixed integer programming hull)
Let $P = \{x \in \mathbb{R}^{k+l} : Ax \leq b\}$ be a rational polyhedron. Show that $\text{conv}(P \cap (\mathbb{Z}^k \times \mathbb{R}^l))$ is a rational polyhedron. (5 Points)

Exercise 9.4:
The MAXIMUM-STABLE-SET-PROBLEM is defined as follows. Given a graph G, we are looking for a vertex set $S \subseteq V(G)$ of maximum cardinality $|S|$ such that $\{v, w\} \notin E(G)$ for all $v, w \in S$. A vertex set S with $E(G[S]) = \emptyset$ is called *stable set*. This problem can be modeled by the following ILP:

$$\max \sum_{v \in V(G)} x_v \tag{1}$$

$$s.d. \quad x_v + x_w \leq 1 \qquad \forall \{v, w\} \in E(G) \tag{2}$$

$$x_v \in \{0, 1\} \qquad \forall v \in V(G) \tag{3}$$

Show:

1. There is a class of graphs $(G_n)_{n \in \mathbb{N}}$ with $|V(G_n)| = n$ such that the integrality gap has size $\Omega(|V(G_n)|)$, where (3) is replaced by

   $$0 \leq x_v \leq 1 \quad \forall v \in V(G)$$

   in the linear relaxation.

2. Following inequalities are valid for the MAXIMUM-STABLE-SET-PROBLEM:

   $$\sum_{v \in V(C)} x_v \leq 1 \quad \text{for a clique } C \subseteq G,$$

   $$\sum_{v \in V(H)} x_v \leq \left\lceil \frac{|V(H)| - 1}{2} \right\rceil \quad \text{for a circuit } H \subseteq G,$$

   where a clique C is a complete subgraph of G, i.e. $G[C] \simeq K_{|C|}$. Give examples for both types of inequalities, where the optima of the linear relaxations are reduced (strictly).

(5 Points)

Submission deadline: Tuesday, 17.12.2013, before the lecture.