Linear and Integer Optimization

Exercise Sheet 7

Exercise 7.1: Consider a minimum cost flow problem (G, u, b, c) with $u \in \mathbb{N}^{E(G)}$, $b \in \mathbb{Z}^{V(G)}$, and a fixed order $V(G) = \{v_1, v_2, \ldots, v_n\}$ of the vertices. Define a balance perturbation $\epsilon \in \mathbb{K}^{V(G)}$ by $\epsilon(v_i) = \frac{1}{n}$ for $i = 2, 3, \ldots, n$, $\epsilon(v_1) = -\frac{n-1}{n}$, and consider the modified problem $(G, y, b + \epsilon, c)$.

1. Let (r, T) be a spanning tree solution with $r = v_1$. Let $D(v)$ be the set of vertices $v' \in V(G)$ whose $v' - r$-path through the tree contains v. Note $v \in D(v)$.

   Show that the perturbation $b+\epsilon$ decreases the flow of each downward arc (v, w) by $|D(w)|/n$, and increases the flow of each upward arc (v, w) by $|D(v)|/n$.

   Furthermore, show that the flow value on a tree edge of a feasible spanning tree structure for $(G, u, b + \epsilon, c)$ is non-zero and a multiple of $\frac{1}{n}$. (4 Points)

2. Use part one to show that the network simplex algorithm solves the perturbed problem in pseudo polynomial time, independent of the choice of the entering edge. (2 Points)

3. Show that a spanning tree structure (r, T, L, U) is strongly feasible for (G, u, b, c) if and only if it is strongly feasible for $(G, u, b + \epsilon, c)$. Conclude that the network simplex algorithm has a pseudo polynomial running time, when generating strongly feasible spanning tree solutions, independent of the choice of the entering edge.

   (3 Points)

Exercise 7.2:

1. Let $A \in \mathbb{Q}^{m\times n}$ and $B \in \mathbb{Q}^{n\times p}$ be two matrices. Prove:

$$size(AB) \leq 2(p \cdot size(A) + m \cdot size(B)).$$

(2 Points)

2. Let $A \in \mathbb{Q}^{n \times n}$ be an invertible matrix. Prove:

$$size(A^{-1}) \leq 4n^2 \cdot size(A).$$

(4 Points)

Exercise 7.3: Let $(F_n)_{n \in \mathbb{N}}$ be the sequence of Fibonacci numbers, i.e. $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove:

1. $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$. (2 Points)
2. In each iteration i of the continuous fraction expansion, we have $h_i \geq F_{i+1}$. (1 Point)
3. The continuous fraction expansion with input $\frac{p}{q}$ terminates after $O(\log q)$ iterations. (2 Points)

Submission deadline: Tuesday, 03.12.2013, before the lecture.