Winter semester 2013/14 | Research Institute for Discrete Mathematics
Prof. Dr. S. Held | University of Bonn

Linear and Integer Optimization

Exercise Sheet 5

Exercise 5.1: Solve the following LP using the simplex algorithm with Bland's rule. For each iteration, report the steps BTRAN, PRICING, FTRAN, RATIO-Test and UPDATE.

$$\begin{array}{lrl} \max & -x_1 + 4x_2 & \\ s.d. & x_1 - x_2 & \leq 2 \\ & -x_1 + x_2 & \leq 1 \\ & x_1, x_2 & \geq 0 \end{array}$$

(4 Points)

Exercise 5.2: Consider an LP $\max\{c^\intercal x : Ax = b, x \geq 0\}$. Let B be an optimum basis with basic solution $x^\star$ and reduced costs $z_N \leq 0$, e.g. at termination of the simplex algorithm in the PRICING step. Let $I \subseteq N$ be the set of non-basic variables with reduced costs $z_j = 0$ ($j \in I$). Prove:

1. $I = \emptyset \Rightarrow x^\star$ is a unique optimum solution.
2. $x^\star$ is the unique optimum solution if and only if the following LP has the optimum solution value 0:

$$\begin{array}{ll} \max & \sum_{i \in I} x_i \\ s.d. & Ax = b, \\ & x_i = 0, \quad i \in N \setminus I, \\ & x_i \geq 0, \quad i \in B \cup I. \end{array}$$

(4 Points)

Exercise 5.3: (Job Assignment Problem)
In the job assignment problem, n jobs with execution times of $t_1, \ldots, t_n \in \mathbb{R}_+$ need to be processed by m workers. However not every worker can perform every job. For each job $i \in \{1, \ldots, n\}$ a set $S_i \subseteq \{1, \ldots, m\}$ specifies the workers that can perform job i. Several workers can process a job in parallel to speed up the processing time, but each worker can only work at a single job at the same time.

1. Formulate an LP minimizing the total execution time for processing all jobs.
2. Determine the dual LP.
3. Develop a simple polynomial time algorithm for $n = 2$ and $t_1, t_2 > 0$ that finds a primal and dual optimal solution and prove its correctness.

(5 Points)

Exercise 5.4: (Dual Simplex Algorithm)
Let $\max\{c^\intercal x : Ax = b, x \geq 0\}$ be an LP, that is not unbounded and where A has full row rank.
Consider the following algorithm that has as an input the LP and a dual feasible basis B, i.e. $z_N = c_N - A_N^\intercal A_B^{-\intercal} c_B \leq 0$.

BTRAN:

Solve $A_B x_B = b$

PRICING:

If $x_B \geq 0$,**Stop**. Else choose an $i \in \{1, \ldots, m\}$ mit $x_{B_i} < 0$.

FTRAN:

Solve $A_B^\intercal w = e_i$ and compute $\alpha_N = A_N^\intercal w$.

RATIO-Test:

If $\alpha_N \geq 0$, **Stop**. Else choose a

$$j = \arg\min\left\{\frac{z_k}{\alpha_k} : \alpha_k < 0, k \in N\right\}, \text{ and set } \gamma = \frac{z_j}{\alpha_j}.$$

Update:

$$z_N \leftarrow z_N - \gamma\alpha_N, \qquad z_{B_i} \leftarrow -\gamma,$$
$$N \leftarrow N \setminus \{j\} \cup \{B_i\}, \qquad B_i \leftarrow j \quad (\text{now } B \leftarrow B \setminus \{B_i\} \cup \{j\}).$$

Goto **BTRAN**.

Prove

1. If the algorithm stops in the PRICING step, then B is an optimum basis and x_B with $x_N = 0$ is an optimum basic solution. (2 Points)
2. If the algorithm stops in the RATIO-Test, then $P^=(A, b) = \emptyset$. (2 Points)
3. The UPDATE-step transforms a dual feasible basis B into a new basis that is dual feasible again. (3 Points)

Submission deadline: Tuesday, 19.11.2013, before the lecture.