Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization

Exercise Sheet 4

Exercise 4.1: Let C be a convex cone and $-C$ the cone $\{x : -x \in C\}$. We call $L = (C \cap -C)$ the *lineality space* of C.

a) Prove that $\bar{C} := C \cap L^{\perp}$, where $L^{\perp} = \{u \ : \ u^{\intercal} x = 0 \ \forall x \in L\}$, is a pointed cone and that C is the direct sum of its lineality space L and the pointed cone $\bar{C}$, i.e.

$$C = (C \cap L^{\perp}) \oplus L.$$

(2 Points)

b) Show that each polyhedron P has a decomposition $P = (Q + C) \oplus L$, where Q is a polytope, C a pointed cone and L a linear subspace.

(3 Points)

Exercise 4.2: Proof the Birkhoff-von Neumann Theorem: the extreme points of the set of doubly stochastic matrices

$$M = \left\{ A = (a_{ij})_{1 \le i,j \le n} \in K^{n \times n}_{\ge 0} \mid \sum_{i=1}^{n} a_{i,j_0} = \sum_{j=1}^{n} a_{i_0,j} = 1 \text{ for all } 1 \le i_0, j_0, \le n \right\}$$

are precisely the permutation matrices, i.e. the elements of $M \cap \{0,1\}^{n \times n}$. (5 Points)

Exercise 4.3: Let $H = (V, E)$ be a *hypergraph*, i.e. V is a finite set of vertices and $E \subseteq 2^V$. Furthermore, let $F \subseteq V$ and $x, y : F \to \mathbb{R}$. Provide an LP formulation for the following problem and dualize the LP:
Determine (an extension) $x, y : V \setminus F \to \mathbb{R}$ such that

$$\sum_{e \in E} (\max_{v \in e} x(v) - \min_{v \in e} x(v) + \max_{v \in e} y(v) - \min_{v \in e} y(v))$$

is minimized.
Remark: This is a relaxation of the placement problem in chip design. The vertices correspond to connected modules that must be placed minimizing the length of all

interconnects (hyperedges). Vertices in F are preplaced. The problem becomes much harder when requiring disjointness of the modules. (5 Points)

Exercise 4.4: Let G be a directed graph with edge weigths $c : E(G) \to \mathbb{R}_+$. Let $E_1, E_2 \subseteq E(G)$ and $s, t \in V(G)$ with $s \neq t$. Consider the following LP:

$$\begin{array}{lll} \min & \sum\limits_{e \in E(G)} c(e) y_e & \\ s.t. & y_e \leq 0 & \forall e \in E_1, \\ & y_e \geq 0 & \forall e \in E_2, \\ & y_e \geq z_w - z_v & \forall e = (v, w) \in E(G), \\ & z_t - z_s = 1. & \end{array}$$

Show:

1. The LP has a solution if and only if every s-t-path in G contains an edge from $E(G) \setminus E_1$. (2 Points)
2. If the LP has an optimum solution $(\tilde{y}, \tilde{z})$, then there is a set X with $X \cap \{s, t\} = \{s\}$ and an optimum solution (y, z) such that $y_e = 1$ for all $e \in \delta^+(X)$, $y_e = -1$ for all $e \in \delta^-(X) \setminus E_2$, and $y_e = 0$ for all remaining edges.
Hint: Consider the set $\{v \in V(G) : \tilde{z}_v \leq \tilde{z}_s\}$ and apply the complementary slackness constraints. (3 Points)

Submission deadline: Tuesday, 12.11.2013, before the lecture.