Linear and Integer Optimization

Exercise Sheet 1

Exercise 1.1:
A paper mill produces paper rolls of 3 m width. The customers order rolls with smaller widths and the mill has to cut the ordered rolls out of the 3 m wide rolls. For example, a 3 m wide roll may be cut into two 93 cm wide and a 108 cm wide roll, leaving an offcut of 6 cm.
The current order consists of

- 90 rolls of width 130 cm,
- 610 rolls of width 108 cm,
- 395 rolls of width 42 cm, and
- 211 rolls of width 93 cm.

Formulate a linear program that minimizes the number of produced 3 m rolls and allows a correct cutting of the ordered rolls.

(4 points)

Exercise 1.2:
Let $A = \begin{pmatrix} -1 & -1 & 1 & 0 & 1 & 1 & -2 \\ 1 & 2 & 1 & 1 & -2 & 0 & -10 \end{pmatrix}^\intercal$ and $b = \begin{pmatrix} 1 & 4 & 8 & 4 & 2 & 5 & -11 \end{pmatrix}^\intercal$. Solve the LP $\max\{c^\intercal x \;:\; Ax \le b\}$ (graphically) and specify the set of optimum solutions for following cost vectors:

1. $c = \begin{pmatrix} 0 & -3 \end{pmatrix}^\intercal$
2. $c = \begin{pmatrix} 1 & 2 \end{pmatrix}^\intercal$
3. $c = \begin{pmatrix} 1 & -2 \end{pmatrix}^\intercal$

Let $P := P(A, b) = \{x \in \mathbb{R}^n \;:\; Ax \le b\}$. How many null-, one-, und two-dimensional faces does P have?

Give an example of a face for each of the three dimensions in the form $F = \{x \in P \ : \ A'x = b'\}$. Here $A'x \leq b'$ is a subsystem of $Ax \leq b$. (4 Points)

Exercise 1.3: Specify necessary and sufficient conditions for the numbers $\alpha, \beta, \gamma \in \mathbb{K}$ so that the LP $\max\{x + y \ : \ \alpha x + \beta y \leq \gamma; \ x, y \geq 0\}$

- has an optimum solution;
- has a feasible solution;
- is unbounded.

(4 Points)

Exercise 1.4: The dimension of a non-empty set $X \subseteq \mathbb{K}^n$ is the number

$$\dim X := n - \max\{\text{rank}(A) \ : \ A \text{ is an } n \times n \text{ matrix with } Ax = Ay \ \forall x, y \in X\}.$$

X is called **full-dimensional** if $\dim X = n$.
Prove: A polyhedron is full-dimensional if and only if there is a point in its interior.
(8 Points)

Submission deadline: Tuesday, 22.10.2013, before the lecture (in groups of 2 students).