Winter term 2014/15
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Combinatorial Optimization

Exercise Sheet 14

Exercise 14.1:
Let U be a finite set and $f : 2^U \to \mathbb{R}$. Define extensions $f^+, f^- : [0,1]^U \to \mathbb{R}$ by

$$f^+(x) := \max\{\sum_{A\subseteq U} \lambda_A f(A) : \sum_{A\subseteq U} \lambda_A 1_A = x, \sum_{A\subseteq U} \lambda_A = 1, \lambda_A \geq 0\}$$

and

$$f^-(x) := \min\{\sum_{A\subseteq U} \lambda_A f(A) : \sum_{A\subseteq U} \lambda_A 1_A = x, \sum_{A\subseteq U} \lambda_A = 1, \lambda_A \geq 0\}.$$

The function f^+ is called concave closure of f, the function f^- is called convex closure of f.

(1) Prove that f^+ is concave and f^- is convex. (1 Point)

(2) Prove that the Lovász extension f' as defined in Exercise 13.3 and the convex closure f^- are identical if and only if f is submodular. (3 Points)

Exercise 14.2:
Let U be a finite set and $f : 2^U \to \mathbb{R}$ be a submodular function. Let R be a random subset of U, where each element is chosen independently with probability $\frac{1}{2}$. Prove:

(1) $\mathrm{Exp}(f(R)) \geq \frac{1}{2}(f(\emptyset) + f(U))$.

(2) For each $A \subseteq U$ we have $\mathrm{Exp}(f(R)) \geq \frac{1}{4}(f(\emptyset) + f(A) + f(U \setminus A) + f(U))$.
Hint: Apply (1) twice.

(3) If f is nonnegative, then $\mathrm{Exp}(f(R)) \geq \frac{1}{4} \max_{A\subseteq U} f(A)$.

Note: Part (3) implies a randomized 4-factor approximation algorithm for nonnegative submodular function maximization. (4 Points)

Exercise 14.3:
Let U be a finite set. A function $f : 2^U \to \mathbb{R} \cup \{\infty\}$ is called crossing submodular if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ for any two sets $X, Y \subseteq U$ with $X \cap Y \neq \varnothing$ and $X \cup Y \neq U$. The SUBMODULAR FLOW PROBLEM is as follows. Given a digraph G, functions $l : E(G) \to \mathbb{R} \cup \{-\infty\}$, $u : E(G) \to \mathbb{R} \cup \{\infty\}$, $c : E(G) \to \mathbb{R}$ and a crossing submodular function $b : 2^{V(G)} \to \mathbb{R} \cup \{\infty\}$, a feasible submodular flow is a function $f : E(G) \to \mathbb{R}$ with $l(e) \leq f(e) \leq u(e)$ for all $e \in E(G)$ and

$$\sum_{e \in \delta^-(X)} f(e) - \sum_{e \in \delta^+(X)} f(e) \leq b(X)$$

for all $X \subseteq V(G)$. The task is to decide whether a feasible flow exists, and if yes, find one whose cost $\sum_{e \in E(G)} c(e) f(e)$ is minimum.
Show that this problem generalizes the MINIMUM COST FLOW PROBLEM and the problem of optimizing a linear function over the intersection of two polymatroids.
(4 Points)

Exercise 14.4:
Given an instance (G, c, r) of the SURVIVABLE NETWORK DESIGN PROBLEM, where G is an undirected graph, $c : E(G) \to \mathbb{R}_{\geq 0}$ is a cost function on the edges and $r_{\{x,y\}} \in \mathbb{Z}_{\geq 0}$ is a connectivity requirement for each pair $\{x, y\} \subseteq V(G)$, define $f : 2^{V(G)} \to \mathbb{Z}_{\geq 0}$ by $f(\varnothing) := f(V(G)) := 0$ and $f(S) := \max_{x \in S, y \in V(G) \setminus S} r_{\{x,y\}}$ for $\varnothing \neq S \subset V(G)$. Then the SURVIVABLE NETWORK DESIGN PROBLEM can be formulated as

$$\min\{\sum_{e \in E(G)} c(e) x_e : \sum_{e \in \delta(S)} x_e \geq f(S) \ \forall S \subseteq V(G),\ x \in \{0,1\}^{E(G)}\}.$$

Show that the relaxation

$$\min\{\sum_{e \in E(G)} c(e) x_e : \sum_{e \in \delta(S)} x_e \geq f(S) \ \forall S \subseteq V(G),\ x \in [0,1]^{E(G)}\}$$

can be reformulated as a linear program of polynomial size. (4 Points)

Deadline: Tuesday, January 27, 2015, before the lecture.
Information: Submissions by groups of one or two students are allowed. This is the last exercise sheet.