Winter term 2014/15
Professor Dr. Stephan Held
Jannik Silvanus

Research Institute for Discrete Mathematics
University of Bonn

Combinatorial Optimization

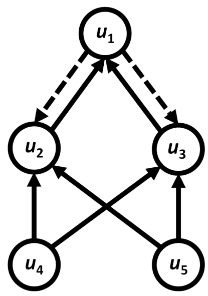
Exercise Sheet 13

Exercise 13.1:
Let E be a finite set and $f : 2^E \to \mathbb{R}$. Prove that f is submodular if and only if $f(T \cup \{e\}) - f(T) \leq f(S \cup \{e\}) - f(S)$ for all $S \subset T \subset T \cup \{e\} \subseteq E$. (4 Points)

Exercise 13.2:
Prove that the set of vertices of the base polyhedron of a submodular function $f : 2^E \to \mathbb{R}$ with $f(\varnothing) = 0$ is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of E, where $b^{\prec}(e) := f(\{v \in E \,|\, v \preceq e\}) - f(\{v \in E \,|\, v \prec e\}))$ for all $e \in E$. (4 Points)

Exercise 13.3:
Let S be a finite set and $f : 2^S \to \mathbb{R}$. Define $f' : \mathbb{R}^S_{\geq 0} \to \mathbb{R}$ as follows. For any $x \in \mathbb{R}^S_{\geq 0}$, there are unique $k \in \mathbb{Z}_{\geq 0}, \lambda_1, \ldots, \lambda_k > 0$ and $\varnothing \subset T_1 \subset T_2 \subset \ldots \subset T_k \subseteq S$ such that $x = \sum_{i=1}^k \lambda_i \chi^{T_i}$, where χ^{T_i} is the incidence vector of T_i. Then $f'(x) := \sum_{i=1}^k \lambda_i f(T_i)$. Prove that f is submodular if and only if f' is convex. (4 Points)

Exercise 13.4:
Prove that for each $\epsilon > 0$, there is a finite set E_ϵ and a submodular function $f_\epsilon : 2^{E_\epsilon} \to \mathbb{R}_{\geq 0}$ such that the Deterministic Double Greedy Algorithm for USM (Algorithm 12 from the lecture) provides only a $(\frac{1}{3} + \epsilon)$-approximation on (E_ϵ, f_ϵ).
Hint: Consider the cut function of the weighted graph on the right. The weight of dashed edges is $1 - \epsilon$, the weight of all other edges is 1. (4 Points)



Deadline: Tuesday, January 20, 2015, before the lecture.
Information: Submissions by groups of one or two students are allowed.