Winter term 2014/15
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Combinatorial Optimization

Exercise Sheet 11

Exercise 11.1: Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid.

(1) Let $X \subseteq E$. Prove: Let Y_1 be a base of $\mathcal{M} \setminus X$ and Y_2 be a base of $\mathcal{M}/(E \setminus X)$, then $Y_1 \cup Y_2$ is a base of $\mathcal{M}$ ($\mathcal{M} \setminus X$ is the matroid $(E \setminus X, \{M \subseteq E \setminus X | M \in \mathcal{F}\})$). (2 Points)

Now let N and K be nonempty subsets of E. A game $\langle \mathcal{M}; N, K \rangle$ is played as follows: Angelika and Bodo (who plays first) alternatingly tag different elements of N. A tagged element cannot be tagged again later in the game. Angelika wins if she tags a set of elements that span K. Bodo wins if all elements of N are tagged and Angelika did not win.

(2) Prove: If N contains two disjoint subsets A_0 and B_0 which span each other and which both span K, then Angelika can win against any strategy Bodo might have (a subset U spans a subset V if $V \subseteq \sigma(U)$).
Hint: Assume Bodo picks $a_0 \in A_0$, then there is a $b_0 \in B_0$ such that $(A_0 \setminus \{a_0\}) \cup \{b_0\}$ is a base of $\mathcal{M}_0 := \mathcal{M} \setminus \sigma(A_0 \cup B_0)$... Also, Exercise 10.3 might be helpful. (4 Points)

Note: The other direction is also true, but harder to prove. You may use this fact for the next exercise.
Now Angelika and Bodo play the even funnier game $\langle G; u, v \rangle$. Here G is a graph and u and v are vertices of G. Angelika and Bodo alternatingly tag edges. Angelika wins if her edges contain a u-v-path and Bodo wins if Angelika didn't win when all edges are tagged. Again Bodo plays first.

(3) Prove: Angelika has a winning strategy if and only if there are $V' \subseteq V(G)$, $E_1 \subseteq E(G)$, and $E_2 \subseteq E(G)$ with $\{u, v\} \subseteq V'$ and $E_1 \cap E_2 = \varnothing$ such that (V', E_1) and (V', E_2) are trees. (2 Points)

Exercise 11.2: Let P be a nonempty polymatroid. Show that there is a monotone function f with $f(\varnothing) = 0$ and $P = P(f)$. (4 Points)

Exercise 11.3: Let G be a directed graph, $s, t \in V(G)$, $u : E(G) \to \mathbb{R}_+$, and $A := \delta^+(s)$. Prove that

$$P := \{x \in \mathbb{R}_+^A \,|\, \text{there is an } s\text{-}t\text{-flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in A\}$$

is a polymatroid. (4 Points)

Deadline: Tuesday, December 23, 2014, before the lecture.
Information: Submissions by groups of one or two students are allowed.